Sample Problems

Compute each of the following integrals.

1. $\int e^{-4x}\,dx$
2. $\int_0^8 e^{-4x}\,dx$
3. $\int \left(x^2-2\right)\left(x^3-6x\right)^{207}\,dx$
4. $\int \frac{(\ln x)^8+1}{x}\,dx$
5. $\int_1^{e^2} \frac{(\ln x)^8+1}{x}\,dx$
6. $\int \frac{3x}{\left(x^2+1\right)^7}\,dx$
7. $\int \frac{12x^3}{3x^4+1}\,dx$
8. $\int (-3x+4)\,e^{-3x^2+8x}\,dx$
9. $\int_0^2 (-3x+4)\,e^{-3x^2+8x}\,dx$
10. $\int \frac{20x}{\left(x^2+1\right)^{20}}\,dx$
11. $\int \frac{e^x}{e^x+1}\,dx$
12. $\int \frac{1}{\sqrt{2x-1}}\,dx$
13. $\int \frac{x}{\sqrt{x+1}}\,dx$
14. $\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$
15. $\int_0^3 \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$
16. $\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx$
17. $\int_0^1 \frac{x^2-8x+3}{x+2}\,dx$
18. $\int_0^\infty xe^{-3x^2}\,dx$
19. $\int_0^\infty 3e^{-3x}\,dx$
20. $\int \frac{2x+5}{\sqrt{x-3}}\,dx$

Practice Problems

Compute each of the following integrals.

1. $\int 200x\left(2x^2-5\right)^{99}\,dx$
2. $\int \frac{e^x}{\left(e^x+2\right)^3}\,dx$
3. $\int \frac{e^x}{e^x+1}\,dx$
4. $\int_0^1 \frac{x+1}{x^2+2x+3}\,dx$
5. $\int \sqrt{3x-2}\,dx$
6. $\int \frac{x^2}{\left(x^3+5\right)^{10}}\,dx$
7. $\int \frac{1}{x\ln x}\,dx$
8. $\int x^3\,2^{x^4-1}\,dx$
9. $\int_{-1}^1 (x+2)\left(x^2+4x-3\right)^3\,dx$
10. $\int \frac{x-1}{\sqrt{x+1}}\,dx$
11. $\int_3^8 \frac{x-1}{\sqrt{x+1}}\,dx$
12. $\int_1^3 \frac{1}{3x-2}\,dx$

Sample Problems - Answers

1.) $-\frac{1}{4}e^{-4x}+C$ 2.) $\frac{1}{4}-\frac{1}{4}e^{-32}$ 3.) $\frac{1}{624}\left(x^3-6x\right)^{208}+C$ 4.) $\frac{\ln^9 x}{9}+\ln x+C$

5.) $\frac{530}{9}$ 6.) $-\frac{1}{4\left(x^2+1\right)^6}+C$ 7.) $\ln\left(3x^4+1\right)+C$ 8.) $\frac{1}{2}e^{-3x^2+8x}+C$

9.) $\frac{1}{2}e^4-\frac{1}{2}$ 10.) $-\frac{10}{19}\left(x^2+1\right)^{-19}+C$ 11.) $\ln\left(e^x+1\right)+C$ 12.) $\sqrt{2x-1}+C$

13.) $\frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}+C$ 14.) 36 15.) $\ln\left(e^3+\frac{1}{e^3}\right)-\ln 2$

16.) $2\ln\left(\sqrt{x}+1\right)+C$ 17.) $23\ln 3-23\ln 2-\frac{19}{2}$ 18.) $\frac{1}{6}$ 19.) 1

20.) $\frac{4}{3}(x-3)^{3/2}+22(x-3)^{1/2}+C$

Practice Problems - Answers

1.) $\frac{1}{2}\left(2x^2-5\right)^{100}+C$ 2.) $-\frac{1}{2\left(e^x+2\right)^2}+C$ 3.) $\ln\left(e^x+1\right)+C$ 4.) $\frac{1}{2}\ln 2$

5) $\frac{2}{9}(3x-2)^{3/2}+C$ 6.) $-\frac{1}{27\left(x^3+5\right)^9}+C$ 7.) $\ln|\ln x|+C$ 8.) $\frac{2^{x^4-1}}{4\ln 2}+C$

9.) -160 10.) $\frac{2}{3}(x+1)^{3/2}-4\sqrt{x+1}+C$ 11.) $\frac{26}{3}$ 12.) $\frac{1}{3}\ln 7$

Sample Problems - Solutions

Compute each of the following integrals.

1. $\displaystyle\int e^{-4x}\,dx$

   Solution: Let $u=-4x$. Then $du=-4dx$ and so $dx=-\frac{1}{4}du$. We now substitute in the integral

$$\int e^{-4x}\,dx=\int e^{u}\;-\frac{1}{4}du=-\frac{1}{4}\int e^{u}du=-\frac{1}{4}e^{u}+C=-\frac{1}{4}e^{-4x}+C$$

2. $\displaystyle\int_0^8 e^{-4x}\,dx$

   Solution:

$$\int_0^8 e^{-4x}\,dx=-\frac{1}{4}e^{-4x}\Big|_0^8=-\frac{1}{4}\left(e^{-4(8)}-e^{-4(0)}\right)=-\frac{1}{4}\left(e^{-32}-e^{0}\right)=-\frac{1}{4}\left(e^{-32}-1\right)=\frac{1}{4}-\frac{1}{4e^{32}}$$

3. $\displaystyle\int\left(x^2-2\right)\left(x^3-6x\right)^{207}\,dx$

   Solution: Let $u=x^3-6x$. Then $du=\left(3x^2-6\right)dx$ and so $dx=\dfrac{1}{(3x^2-6)}du$. We now substitute in the integral

$$\int\left(x^2-2\right)\left(x^3-6x\right)^{207}\,dx=$$

$$=\int\left(x^2-2\right)u^{207}\,\frac{1}{(3x^2-6)}du=\int\left(x^2-2\right)u^{207}\,\frac{1}{3\left(x^2-2\right)}du=\frac{1}{3}\int u^{207}\,du$$

$$=\frac{1}{3}\left(\frac{u^{208}}{208}\right)+C=\frac{1}{624}u^{208}+C=\frac{1}{624}\left(x^3-6x\right)^{208}+C$$

4. $\displaystyle\int\frac{(\ln x)^8+1}{x}\,dx$

   Solution: Let $u=\ln x$. Then $du=\frac{1}{x}dx$ and so $dx=xdu$.

$$\int\frac{(\ln x)^8+1}{x}\,dx=\int\frac{u^8+1}{\not x}\,\not x\,du=\int\left(u^8+1\right)du=\frac{u^9}{9}+u+C=\frac{(\ln x)^9}{9}+\ln x+C$$

5. $\displaystyle\int_1^{e^2}\frac{(\ln x)^8+1}{x}\,dx$

   Solution:

$$\int_1^{e^2}\frac{(\ln x)^8+1}{x}\,dx=\frac{(\ln x)^9}{9}+\ln x\Big|_1^{e^2}=\left(\frac{\left(\ln\left(e^2\right)\right)^9}{9}+\ln\left(e^2\right)\right)-\left(\frac{(\ln 1)^9}{9}+\ln 1\right)=\frac{2^9}{9}+2-0=\frac{530}{9}$$

6. $\displaystyle\int \frac{3x}{(x^2+1)^7}\,dx$

Solution: Let $u = x^2+1$. Then $du = 2x dx$ and so $dx = \frac{1}{2x} du$.

$$\int \frac{3x}{(x^2+1)^7}\,dx = \int \frac{3x}{u^7}\,\frac{1}{2x}du = \frac{3}{2}\int \frac{1}{u^7}\,du = \frac{3}{2}\int u^{-7}\,du = \frac{3}{2}\frac{u^{-6}}{-6} + C = -\frac{1}{4u^6} + C$$
$$= -\frac{1}{4(x^2+1)^6} + C$$

7. $\displaystyle\int \frac{12x^3}{3x^4+1}\,dx$

Solution: Let $u = 3x^4+1$. Then $du = 12x^3 dx$ and so $dx = \frac{1}{12x^3} du$.

$$\int \frac{12x^3}{3x^4+1}\,dx = \int \frac{12x^3}{u}\,\frac{1}{12x^3}du = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left(3x^4+1\right) + C$$

8. $\displaystyle\int (-3x+4)\,e^{-3x^2+8x}\,dx$

Solution: Let $u = -3x^2+8x$. Then $du = (-6x+8)\,dx$ and so $dx = \frac{1}{(-6x+8)} du$.

$$\int (-3x+4)\,e^{-3x^2+8x}\,dx = \int (-3x+4)\,e^u\,\frac{1}{(-6x+8)}du = \int (-3x+4)\,e^u\,\frac{1}{2(-3x+4)}du$$
$$= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-3x^2+8x} + C$$

9. $\displaystyle\int_0^2 (-3x+4)\,e^{-3x^2+8x}\,dx$

Solution:

$$\int_0^2 (-3x+4)\,e^{-3x^2+8x}\,dx = \frac{1}{2}e^{-3x^2+8x}\Big|_0^2 = \frac{1}{2}\left(e^{-3(2)^2+8(2)} - e^{-3(0)^2+8(0)}\right) = \frac{1}{2}\left(e^4 - e^0\right) = \frac{1}{2}\left(e^4-1\right)$$

10. $\displaystyle\int \frac{20x}{(x^2+1)^{20}}\,dx$

Solution: Let $u = x^2+1$. Then $du = 2x dx$ and so $dx = \frac{1}{2x} du$

$$\int \frac{20x}{(x^2+1)^{20}}\,dx = \int \frac{20x}{u^{20}}\,\frac{1}{2x}du = 10\int u^{-20}du = 10\left(\frac{u^{-19}}{-19}\right) + C = -\frac{10}{19}\left(x^2+1\right)^{-19} + C$$

11. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

Solution: Let $u = e^x+1$. Then $du = e^x dx$ and so $dx = \frac{1}{e^x} du$.

$$\int \frac{e^x}{e^x+1}\,dx = \int \frac{e^x}{u}\,\frac{1}{e^x}du = \int \frac{1}{u}du = \ln|u| + C = \ln\left(e^x+1\right) + C$$

12. $\displaystyle\int \frac{1}{\sqrt{2x-1}}\,dx$

Solution: Let $u = 2x-1$. Then $du = 2dx$ and so $dx = \frac{1}{2}du$.

$$\int \frac{1}{\sqrt{2x-1}}\,dx = \int \frac{1}{\sqrt{u}}\,\frac{1}{2}du = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left(2u^{1/2}\right) + C = \sqrt{u} + C = \sqrt{2x-1} + C$$

13. $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

Solution: Let $u = x+1$. Then $du = dx$ and also, $x = u-1$.

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\,dx &= \int \frac{u-1}{\sqrt{u}}du = \int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\,du = \int \sqrt{u} - \frac{1}{\sqrt{u}}\,du = \int u^{1/2} - u^{-1/2}\,du \\ &= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C\end{aligned}$$

14. $\displaystyle\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$

Solution:

$$\begin{aligned}\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx &= \left(\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2}\right)\Big|_0^{15} \\ &= \left(\frac{2}{3}(15+1)^{3/2} - 2(15+1)^{1/2}\right) - \left(\frac{2}{3}(0+1)^{3/2} - 2(0+1)^{1/2}\right) \\ &= \left(\frac{2}{3}\left(16^{3/2}\right) - 2\left(16^{1/2}\right)\right) - \left(\frac{2}{3}\left(1^{3/2}\right) - 2\left(1^{1/2}\right)\right) \\ &= \left(\frac{2}{3}(64) - 2(4)\right) - \left(\frac{2}{3} - 2\right) = \frac{128}{3} - 8 - \frac{2}{3} + 2 = \frac{126}{3} - 6 = 36\end{aligned}$$

15. $\displaystyle\int_0^{3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Solution: Let $u = e^x + e^{-x}$. Then $du = (e^x - e^{-x})\,dx$ and so $dx = \frac{1}{(e^x - e^{-x})}du$. Also, when $x = 0$, then $u = e^0 + e^{-0} = 2$ and when $x = 3$, then $u = e^3 + e^{-3}$

$$\int_0^{3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int_2^{e^3+e^{-3}} \frac{e^x - e^{-x}}{u}\,\frac{1}{(e^x - e^{-x})}du = \int_2^{e^3+e^{-3}} \frac{1}{u}\,du = \ln|u|\Big|_2^{e^3+e^{-3}} = \ln\left(e^3 + e^{-3}\right) - \ln 2$$

16. $\displaystyle\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx$

Solution: Let $u = \sqrt{x} + 1$. Then $du = \frac{1}{2\sqrt{x}}dx$ and so $dx = 2\sqrt{x}du$.

$$\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx = \int \frac{1}{\sqrt{x}\;u}\,2\sqrt{x}du = 2\int \frac{1}{u}\,du = 2\ln|u| + C = 2\ln\left(\sqrt{x}+1\right) + C$$

17. $\displaystyle\int_0^1 \frac{x^2-8x+3}{x+2}\,dx$

Solution: We will first work out the indefinite integral. We perform the division betweeen the polynomials first

$$\frac{x^2-8x+3}{x+2} = x-10+\frac{23}{x+2}$$

so the integral is

$$\int \frac{x^2-8x+3}{x+2}\,dx = \int x-10+\frac{23}{x+2}\,dx = \int x-10\,dx + \int \frac{23}{x+2}\,dx = \int x-10\,dx + 23\int \frac{1}{x+2}\,dx$$

The two integrals will be computed using different methods. Clearly,

$$\int x-10\,dx = \frac{x^2}{2}-10x+C_1$$

The integral $\displaystyle\int \frac{1}{x+2}\,dx$ can be computed via substitution. Let $u=x+2$. Then $du=dx$.

$$23\int \frac{1}{x+2}\,dx = 23\int \frac{1}{u}\,du = 23\ln|u|+C_2 = 23\ln|x+2|+C_2$$

The entire integral is

$$\begin{aligned}\int \frac{x^2-8x+3}{x+2}\,dx &= \frac{x^2}{2}-10x+C_1+23\ln|x+2|+C_2\\ &= \frac{x^2}{2}-10x+23\ln|x+2|+C\end{aligned}$$

Then the definite integral is.

$$\begin{aligned}\int_0^1 \frac{x^2-8x+3}{x+2}\,dx &= \left(\frac{x^2}{2}-10x+23\ln|x+2|\right)\Bigg|_0^1\\ &= \left(\frac{1^2}{2}-10(1)+23\ln|1+2|\right)-\left(\frac{0^2}{2}-10(0)+23\ln|0+2|\right)\\ &= -\frac{19}{2}+23\ln 3-23\ln 2 \approx -0.174\,302\,51\end{aligned}$$

18. $\displaystyle\int_0^\infty xe^{-3x^2}\,dx$

Solution: We will first work out the indefinite integral. Let $u=-3x^2$. Then $du=-6x\,dx$ and so $dx=-\dfrac{1}{6x}du$.

$$\int xe^{-3x^2}\,dx = \int xe^u\ -\frac{1}{6x}du = -\frac{1}{6}\int e^u\,du = -\frac{1}{6}e^u+C = -\frac{1}{6}e^{-3x^2}+C$$

To compute the improper integral, we take the limit of definite integrals.

$$\begin{aligned}\int_0^\infty xe^{-3x^2}\,dx &= \lim_{N\to\infty}\int_0^N xe^{-3x^2}\,dx = \lim_{N\to\infty}\left(-\frac{1}{6}e^{-3x^2}\Bigg|_0^N\right) = \lim_{N\to\infty}\left(-\frac{1}{6}\left(e^{-3N^2}-e^{-3(0^2)}\right)\right)\\ &= -\frac{1}{6}\lim_{N\to\infty}\left(e^{-3N^2}-1\right) = \frac{1}{6}\lim_{N\to\infty}\left(1-\frac{1}{e^{3N^2}}\right) = \frac{1}{6}\end{aligned}$$

since $\displaystyle\lim_{N\to\infty}\frac{1}{e^{3N^2}}=0$.

19. $\displaystyle\int_0^{\infty} 3e^{-3x}\ dx$

Solution: We will first work out the indefinite integral. Let $u=-3x$. Then $du=-3dx$ and so $dx=-\frac{1}{3}du$.

$$\int 3e^{-3x}\ dx=\int 3e^{u}\ \left(-\frac{1}{3}\right)du=-\int e^{u}\ du=-e^{u}+C=-e^{-3x}+C$$

To compute the improper integral, we take the limit of definite integrals.

$$\begin{aligned}\int_0^{\infty} 3e^{-3x}\ dx &= \lim_{N\to\infty}\int_0^{N} 3e^{-3x}\ dx\\ &= \lim_{N\to\infty}\left(-e^{-3x}\Big|_0^N\right)=\lim_{N\to\infty}\left(-\left(e^{-3N}-e^{-3(0)}\right)\right)=\lim_{N\to\infty}\left(1-\frac{1}{e^{3N}}\right)=1\end{aligned}$$

since $\displaystyle\lim_{N\to\infty}\frac{1}{e^{3N}}=0$.

20. $\displaystyle\int\frac{2x+5}{\sqrt{x-3}}\ dx$

Solution: Let $u=x-3$. Then $du=dx$ and also $x=u+3$.

$$\begin{aligned}\int\frac{2x+5}{\sqrt{x-3}}\ dx &= \int\frac{2(u+3)+5}{\sqrt{u}}du=\int\frac{2u+11}{\sqrt{u}}\ du=\int\frac{2u}{\sqrt{u}}+\frac{11}{\sqrt{u}}\ du\\ &= \int 2\sqrt{u}\ du+\int\frac{11}{\sqrt{u}}\ du=2\int u^{1/2}\ du+11\int u^{-1/2}\ du\\ &= 2\left(\frac{2}{3}u^{3/2}\right)+C_1+11\left(2u^{1/2}\right)+C_2\\ &= \frac{4}{3}(x-3)^{3/2}+22(x-3)^{1/2}+C\end{aligned}$$

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