Sample Problems

Compute each of the following integrals.

1. $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$
2. $\displaystyle\int \sqrt{1-x^2}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$
4. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$
5. $\displaystyle\int \sqrt{x^2+4}\,dx$
6. $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$
7. $\displaystyle\int \frac{x^6}{\sqrt{1-x^{14}}}\,dx$
8. $\displaystyle\int_0^1 \frac{\tan^{-1} x}{x^2+1}\,dx$

Practice Problems

Compute each of the following integrals. Please note that some of the integrals can also be solved using other, previously seen methods.

1. $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$
2. $\displaystyle\int \frac{x}{\sqrt{x^2-25}}\,dx$
3. $\displaystyle\int \frac{x^2}{\sqrt{x^2-25}}\,dx$
4. $\displaystyle\int \sqrt{x^2-25}\,dx$
5. $\displaystyle\int \sqrt{16-x^2}\,dx$
6. $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$
7. $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$
8. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$
9. $\displaystyle\int \sqrt{x^2+1}\,dx$
10. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$
11. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$
12. $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

Sample Problems - Answers

1.) $\ln\left|x+\sqrt{x^2+4}\right|+C$ 2.) $\frac{1}{2}\sin^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+C$ 3.) $\ln\left|x+\sqrt{x^2-9}\right|+C$

4.) $8\sin^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}x\sqrt{16-x^2}+C$ 5.) $\frac{1}{2}x\sqrt{x^2+4}+2\ln\left|x+\sqrt{x^2+4}\right|+C$

6.) $\frac{1}{2}x\sqrt{x^2+9}-\frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right|+C$ 7.) $\frac{1}{7}\sin^{-1}\left(x^7\right)+C$ 8.) $\frac{\pi^2}{32}$

Practice Problems - Answers

1.) $\ln\left|x+\sqrt{x^2-25}\right|+C$ 2.) $\sqrt{x^2-25}+C$ 3.) $\frac{1}{2}x\sqrt{x^2-25}+\frac{25}{2}\ln\left|x+\sqrt{x^2-25}\right|+C$

4.) $\frac{1}{2}x\sqrt{x^2-25}-\frac{25}{2}\ln\left|x+\sqrt{x^2-25}\right|+C$ 5.) $8\sin^{-1}\left(\frac{x}{4}\right)+\frac{1}{2}x\sqrt{16-x^2}+C$ 6.) $\sin^{-1}\left(\frac{x}{4}\right)+C$

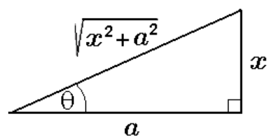
7.) $-\sqrt{16-x^2}+C$ 8.) $8\sin^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}x\sqrt{16-x^2}+C$ 9.) $\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C$

10.) $\ln\left|x+\sqrt{x^2+1}\right|+C$ 11.) $\sqrt{x^2+1}+C$ 12.) $\frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C$

Sample Problems - Solutions

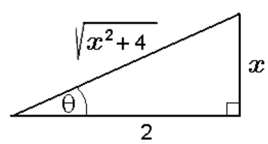
Trigonometric substitution is a technique of integration. It is especially useful in handling expressions under a square root sign.

Case 1. The substitution $x = a\tan\theta$. This is useful in handling an integral involving $\sqrt{x^2+a^2}$.
Let $x = a\tan\theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. (That is the same thing as stating that $\theta = \tan^{-1}\dfrac{x}{a}$. The interval between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ is the domain of the inverse function $\tan^{-1}x$.)
The picture below shows the reference triangle we use for this substitution.



Using this triangle, we do not have to do heavy duty algebra because we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 1: Compute the integral $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$.
Solution: We will use a trigonometric substitution. We start with a reference triangle where the hypotenuse is the denominator. Using the substitution $x = 2\tan\theta$, (where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$) we will transform the integral into one in θ.



From the triangle, $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\ d\theta$. The expression $\sqrt{x^2+4}$ becomes $2\sec\theta$ - using the picture, or using algebra. Recall the identity $\tan^2 x + 1 = \sec^2 x$

$$\sqrt{x^2+4} = \sqrt{(2\tan\theta)^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4}\sqrt{\tan^2\theta+1} = 2\sqrt{\sec^2\theta} = 2\,|\sec\theta|$$

Because θ is in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, $\sec x$ is positive and so $|\sec x| = \sec x$.

$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{1}{2\sec\theta}\ \left(2\sec^2\theta\ d\theta\right) = \int \sec\theta\ d\theta$$

This is an integral we have already seen: we can either use substitution (see in that handout) or partial fraction(see in that handout). Either way,

$$\int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + C$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

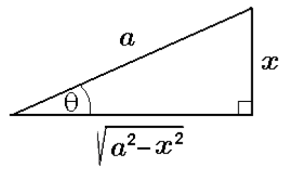
$$\sec\theta = \frac{\sqrt{x^2+4}}{2} \qquad \text{and} \qquad \tan\theta = \frac{x}{2}$$

Thus the answer is $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx = \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$. This expression can be further simplified:

$$\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + C = \ln\left|\sqrt{x^2+4}+x\right| - \ln 2 + C = \ln\left|\sqrt{x^2+4}+x\right| + C$$

and so the final answer is $\boxed{\ln\left|\sqrt{x^2+4}+x\right|+C}$.

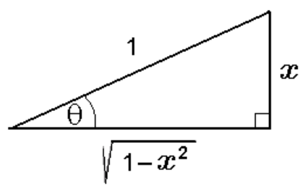
Case 2. The substitution $x = a\sin\theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. This is useful in handling an integral involving $\sqrt{a^2-x^2}$. The picture below shows the reference triangle we use for this substitution.



Using this triangle, we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 2: Compute the integral $\displaystyle\int \sqrt{1-x^2}\,dx$.

Solution: This is a very famous integral because it leads to the area formula of the unit circle. We will use a trigonometric substitution. We start with a reference triangle where the $\sqrt{1-x^2}$ is one of the legs. Using the substitution $x=\sin\theta$, $(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2})$ we will transform the integral into one in θ.



From the triangle, $x=\sin\theta$. Then $dx=\cos\theta\;d\theta$. The expression $\sqrt{1-x^2}$ becomes

$$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=|\cos\theta|$$

Because θ is in the interval $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, $\cos x$ is positive and so $|\cos x|=\cos x$.

$$\int\sqrt{1-x^2}\;dx=\int\cos\theta\;\;(\cos\theta\;d\theta)=\int\cos^2\theta\;d\theta$$

This is an integral we have already seen; we can simplify it using the double angle formula for cosine.

$$\cos 2\theta = 2\cos^2\theta - 1 \qquad\Longrightarrow\qquad \cos^2\theta=\frac{1}{2}\left(\cos 2\theta+1\right)$$

$$\begin{aligned}\int\cos^2\theta\;d\theta &= \int\frac{1}{2}\left(\cos 2\theta+1\right)\;d\theta=\frac{1}{2}\int\cos 2\theta+1\;d\theta=\frac{1}{2}\left(\frac{1}{2}\sin 2\theta+\theta\right)+C\\ &= \frac{1}{2}\left(\frac{1}{2}\left(2\sin\theta\cos\theta\right)+\theta\right)+C=\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta+C\end{aligned}$$

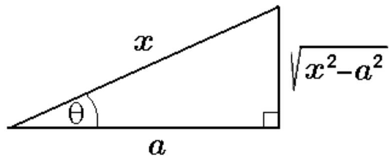
Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

$$\sin\theta=x,\qquad\cos\theta=\sqrt{1-x^2}\qquad\text{and}\qquad\theta=\sin^{-1}x$$

Thus the answer is $\displaystyle\int\sqrt{1-x^2}\;dx=\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta+C=\boxed{\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+C}$

Note that if we now compute $\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\;dx$ the result is the area of the unit semi-circle, $\dfrac{\pi}{2}$.

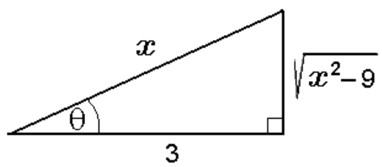
Case 3. The substitution $x = a \sec \theta$ where $0 < \theta < \frac{\pi}{2}$. This is useful in handling an integral involving $\sqrt{x^2 - a^2}$. The picture below shows the reference triangle we use for this substitution.



Using this triangle, we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 3: Compute the integral $\int \frac{1}{\sqrt{x^2 - 9}} \, dx$.

Solution: We will use a trigonometric substitution. We start with a reference triangle where the hypotenuse is x and one shorter side is 3. Using the substitution $x = 3 \sec \theta$, we will transform the integral into one in θ.



From the triangle, $x = 3 \sec \theta$. Then $dx = 3 \sec \theta \tan \theta \, d\theta$. The expression $\sqrt{x^2 - 9}$ becomes $3 \tan \theta$ - either from the picture or using algebra. Recall the identity $\sec^2 x = \tan^2 x + 1$

$$\sqrt{x^2 - 9} = \sqrt{(3 \sec \theta)^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = \sqrt{9}\sqrt{\sec^2 \theta - 1} = 3\sqrt{\tan^2 \theta} = 3 \left| \tan \theta \right|$$

Because $0 < \theta < \frac{\pi}{2}$, $\tan \theta$ is positive and so $\left| \tan \theta \right| = \tan \theta$.

$$\int \frac{1}{\sqrt{x^2 - 9}} \, dx = \int \frac{1}{3 \tan \theta} \, (3 \sec \theta \tan \theta \, d\theta) = \int \sec \theta \, d\theta$$

Again,

$$\int \sec \theta \, d\theta = \ln \left| \sec \theta + \tan \theta \right| + C$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

$$\sec \theta = \frac{x}{3} \quad \text{and} \quad \tan \theta = \frac{\sqrt{x^2 - 9}}{3}$$

Thus the answer is $\int \frac{1}{\sqrt{x^2 - 9}} \, dx = \ln \left| \sec \theta + \tan \theta \right| + C = \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| + C$. We can still simplify this result a bit:

$$\ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| + C = \ln \left| \frac{x + \sqrt{x^2 - 9}}{3} \right| + C = \ln \left| x + \sqrt{x^2 - 9} \right| - \ln 3 + C = \ln \left| x + \sqrt{x^2 - 9} \right| + C_2$$

Thus the final answer is $\int \frac{1}{\sqrt{x^2 - 9}} \, dx = \boxed{\ln \left| x + \sqrt{x^2 - 9} \right| + C}$.

Example 4: Compute the integral $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

Solution: Let $x = 4\sin\theta$ where $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$. Then $dx = 4\cos\theta\ d\theta$ and

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16}\sqrt{1-\sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\left|\cos\theta\right|$$

Because $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, $\cos\theta$ is non-negative, and $|\cos\theta| = \cos\theta$. So the integral is

$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta}{4\cos\theta}\ (4\cos\theta\ d\theta) = \int 16\sin^2\theta\ d\theta = 16\int \sin^2\theta\ d\theta$$

By the double angle formula for cosine, $\cos 2\theta = 1 - 2\sin^2\theta \quad\Longrightarrow\quad \sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= 16\int \sin^2\theta\ d\theta = 16\int \frac{1}{2}(1-\cos 2\theta)\ d\theta = 8\int 1-\cos 2\theta\ d\theta = 8\left(\theta - \frac{1}{2}\sin 2\theta + C\right)\\ &= 8\theta - 4\sin 2\theta + C\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy. Recall that $x = 4\sin\theta$ and so

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{x}{4}\right) \qquad \text{and}\\ \sin 2\theta &= 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2\left(\frac{x}{4}\right)\sqrt{1-\left(\frac{x}{4}\right)^2} = \frac{x}{2}\sqrt{\frac{1}{16}(16-x^2)}\\ &= \frac{x}{2}\left(\frac{1}{4}\right)\sqrt{16-x^2} = \frac{1}{8}x\sqrt{16-x^2}\end{aligned}$$

And so the final answer is $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx = 8\theta - 4\sin 2\theta + C = \boxed{8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C}$

Example 5: Compute the integral $\displaystyle\int \sqrt{x^2+4}\ dx$

Solution: Let $x = 2\tan\beta$ where $-\dfrac{\pi}{2} \le \beta \le \dfrac{\pi}{2}$. Then $dx = 2\sec^2\beta\ d\beta$ and

$$\begin{aligned}\int \sqrt{x^2+4}\ dx &= \int \sqrt{4\tan^2\beta+4}\ \left(2\sec^2\beta\ d\beta\right) = \int 2\sqrt{\tan^2\beta+1}\ \left(2\sec^2\beta\ d\beta\right) = 4\int \left|\sec\beta\right|\ \left(\sec^2\beta\ d\beta\right) =\\ &= 4\int \sec\beta\ \left(\sec^2\beta\ d\beta\right) = 4\int \sec^3\beta\ d\beta\end{aligned}$$

We will compute $\displaystyle\int \sec^3\beta\ d\beta$ by parts.
Let $u = \sec\beta$ and $dv = \sec^2\beta\ d\beta$. Then

$$du = \sec\beta\tan\beta d\beta \quad\text{and}\quad v = \int dv = \int \sec^2\beta\ d\beta$$

In short,

$u = \sec\beta$	$v = \tan\beta$
$du = \sec\beta\tan\beta\ d\beta$	$dv = \sec^2\beta\ d\beta$

$$\begin{aligned}
\int u\ dv &= uv - \int v\ du \text{ becomes}\\
\int \sec\beta\ \sec^2\beta\ d\beta &= \sec\beta\tan\beta - \int \tan\beta\ \sec\beta\tan\beta\ d\beta\\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \tan^2\beta\ \sec\beta\ d\beta \qquad\qquad \text{recall } \tan^2\beta + 1 = \sec^2\beta\\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \left(\sec^2\beta - 1\right)\sec\beta\ d\beta\\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \sec^3\beta - \sec\beta\ d\beta\\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \sec^3\beta\ d\beta + \int \sec\beta\ d\beta\\
2\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta + \int \sec\beta\ d\beta\\
2\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta + \ln|\sec\beta + \tan\beta| + C\\
\int \sec^3\beta\ d\beta &= \frac{1}{2}\sec\beta\tan\beta + \frac{1}{2}\ln|\sec\beta + \tan\beta| + C
\end{aligned}$$

Now the original integral is

$$\begin{aligned}
\int \sqrt{x^2+4}\ dx &= 4\int \sec^3\beta\ d\beta = 4\left(\frac{1}{2}\sec\beta\tan\beta + \frac{1}{2}\ln|\sec\beta+\tan\beta|\right) + C\\
&= 2\sec\beta\tan\beta + 2\ln|\sec\beta+\tan\beta| + C
\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. Recall that $x = 2\tan\beta$. Then $\tan\beta = \dfrac{x}{2}$ and

$$\sec\beta = \sqrt{\tan^2\beta + 1} = \sqrt{\left(\frac{x}{2}\right)^2 + 1} = \sqrt{\frac{1}{4}x^2 + 1} = \sqrt{\frac{1}{4}\left(x^2+4\right)} = \frac{1}{2}\sqrt{x^2+4}$$

and so

$$\begin{aligned}
\int \sqrt{x^2+4}\ dx &= 2\sec\beta\tan\beta + 2\ln|\sec\beta+\tan\beta| + C = 2\left(\frac{1}{2}\sqrt{x^2+4}\right)\left(\frac{x}{2}\right) + 2\ln\left|\frac{1}{2}\sqrt{x^2+4} + \frac{x}{2}\right| + C\\
&= \frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + C = \frac{1}{2}x\sqrt{x^2+4} + 2\left(\ln\left|x+\sqrt{x^2+4}\right| - \ln 2\right) + C\\
&= \frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| - 2\ln 2 + C = \boxed{\frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| + C}
\end{aligned}$$

Example 6: Compute the integral $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$

Let $x = 3\tan\theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Then $dx = 3\sec^2\theta d\theta$ and

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= \int \frac{9\tan^2\theta}{\sqrt{9\tan^2\theta+9}}\left(3\sec^2\theta\ d\theta\right) = \int \frac{9\tan^2\theta}{3\sqrt{\tan^2\theta+1}}\left(3\sec^2\theta\ d\theta\right) = \int \frac{9\tan^2\theta}{3\left|\sec\theta\right|}\left(3\sec^2\theta\ d\theta\right)\\ &= \int \frac{9\tan^2\theta}{3\sec\theta}\left(3\sec^2\theta\ d\theta\right) = 9\int \tan^2\theta\sec\theta\ d\theta = 9\int \left(\sec^2\theta-1\right)\sec\theta\ d\theta\\ &= 9\int \sec^3\theta - \sec\theta\ d\theta = 9\int \sec^3\theta\ d\theta - 9\int \sec\theta\ d\theta\end{aligned}$$

We know that $\displaystyle\int \sec\theta\ d\theta = \ln\left|\sec\theta+\tan\theta\right| + C$ and from the previous computation we have that $\displaystyle\int \sec^3\theta\ d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C$. So that the integral is

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= 9\int \sec^3\theta\ d\theta - 9\int \sec\theta\ d\theta = 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|\right) - 9\ln\left|\sec\theta+\tan\theta\right| + C\\ &= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln\left|\sec\theta+\tan\theta\right| - 9\ln\left|\sec\theta+\tan\theta\right| + C = \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln\left|\sec\theta+\tan\theta\right| + C\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. Recall that $x = 3\tan\theta$. Then $\tan\theta = \dfrac{x}{3}$ and

$$\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{\left(\frac{x}{3}\right)^2+1} = \sqrt{\frac{1}{9}x^2+1} = \sqrt{\frac{1}{9}\left(x^2+9\right)} = \frac{1}{3}\sqrt{x^2+9}$$

and so

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln\left|\sec\theta+\tan\theta\right| + C = \frac{9}{2}\left(\frac{1}{3}\sqrt{x^2+9}\right)\left(\frac{x}{3}\right) - \frac{9}{2}\ln\left|\left(\frac{1}{3}\sqrt{x^2+9}\right)+\frac{x}{3}\right| + C\\ &= \frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + C = \frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\left(\ln\left|x+\sqrt{x^2+9}\right| - \ln 3\right) + C\\ &= \frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + \frac{9}{2}\ln 3 + C = \boxed{\frac{1}{2}x\sqrt{x^2+9} - \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + C}\end{aligned}$$

Example 7: $\displaystyle\int \frac{x^6}{\sqrt{1-x^{14}}}\,dx$

Solution: Let $u = x^7$. Then $du = 7x^6dx$ and so $dx = \dfrac{du}{7x^6}$. Then the integral becomes

$$\int \frac{x^6}{\sqrt{1-u^2}}\,\frac{du}{7x^6} = \frac{1}{7}\int \frac{1}{\sqrt{1-u^2}}\,du$$

We can either recognize that this is the derivative of $\sin^{-1}u$:

$$\frac{1}{7}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{7}\sin^{-1}u + C = \frac{1}{7}\sin^{-1}\left(x^7\right) + C$$

If we do not recognize the derivative, then we can use trigonometric substitution $\theta = \sin^{-1} u$. Then $u = \sin\theta$ and so $du = \cos\theta d\theta$ and θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\frac{1}{7}\int \frac{1}{\sqrt{1-u^2}}\ du = \frac{1}{7}\int \frac{1}{\sqrt{1-\sin^2\theta}}\ \cos\theta d\theta = \frac{1}{7}\int \frac{1}{\sqrt{\cos^2\theta}}\ \cos\theta d\theta = \frac{1}{7}\int \frac{1}{|\cos\theta|}\ \cos\theta d\theta$$

Since θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta$ is non-negative and so $|\cos\theta| = \cos\theta$ and so

$$\frac{1}{7}\int \frac{1}{|\cos\theta|}\ \cos\theta d\theta = \frac{1}{7}\int \frac{1}{\cos\theta}\ \cos\theta d\theta = \frac{1}{7}\int 1 d\theta = \frac{1}{7}\theta + C = \frac{1}{7}\sin^{-1}\left(x^7\right) + C$$

So the answer is $\boxed{\frac{1}{7}\sin^{-1}\left(x^7\right) + C}$.

Example 8: $\displaystyle\int_0^1 \frac{\tan^{-1} x}{x^2+1}\ dx$

Solution: Let $u = \tan^{-1} x$. Then $du = \frac{1}{1+x^2} dx$. For the limits of the integral, when $x = 0$, then $u = \tan^{-1}(0) = 0$ and when $x = 1$, $u = \tan^{-1}(1) = \frac{\pi}{4}$. So our integral becomes

$$\int_0^1 \frac{\tan^{-1} x}{x^2+1}\ dx = \int_0^{\pi/4} u\ du = \left.\frac{u^2}{2}\right|_0^{\pi/4} = \frac{1}{2}\left(\left(\frac{\pi}{4}\right)^2 - 0^2\right) = \frac{1}{2}\cdot\frac{\pi^2}{16} = \boxed{\frac{\pi^2}{32}}$$

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