Sample Problems

1. Compute each of the following limits. Show all steps, using correct notation.

a) $\displaystyle\lim_{x\to\infty}\frac{1}{x}$

b) $\displaystyle\lim_{x\to-\infty}\frac{1}{x}$

c) $\displaystyle\lim_{x\to\infty}\frac{-5}{2x^3}$

d) $\displaystyle\lim_{x\to-\infty}\left(\frac{-5}{2x^3}-7+\frac{8}{x}\right)$

e) $\displaystyle\lim_{x\to\infty}\left(-2x^3+1-\frac{5}{x}+\frac{12}{x^4}\right)$

f) $\displaystyle\lim_{x\to-\infty}\frac{3x-2}{x}$

g) $\displaystyle\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^2}$

h) $\displaystyle\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^3}$

i) $\displaystyle\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^4}$

2. (Polynomials) Compute each of the following limits. Show all steps, using correct notation.

a) $\displaystyle\lim_{x\to-\infty}\left(-2x^5-8x^4+7x^3-10\right)$

b) $\displaystyle\lim_{x\to\infty}\left(-2x^5-8x^4+7x^3-10\right)$

c) $\displaystyle\lim_{x\to-\infty}\left(-2x^5+8x^6\right)$

d) $\displaystyle\lim_{x\to\infty}\left(-2x^5+8x^6\right)$

3. (Rational Functions) Compute each of the following limits. Show all steps, using correct notation.

a) $\displaystyle\lim_{x\to-\infty}\frac{x+x^2-6}{6x+5x^2+2x^3}$

b) $\displaystyle\lim_{x\to\infty}\frac{x^2+9}{5x+2x^2-3}$

c) $\displaystyle\lim_{x\to-\infty}\frac{x^3-9x+1}{3x^2-2x-15}$

4. (More indeterminates) Compute each of the following limits. Show all steps, using correct notation.

a) $\displaystyle\lim_{x\to\infty}\left(3^{x+1}-3^x\right)$

b) $\displaystyle\lim_{x\to\infty}\frac{3\sqrt{x}+2}{5\sqrt{x}+1}$

c) $\displaystyle\lim_{x\to\infty}\left(\sqrt{2x-1}-\sqrt{2x}\right)$

d) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

e) $\displaystyle\lim_{x\to\infty}x\left(\frac{1}{5}-\frac{1}{5-\frac{1}{x}}\right)$

f) $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}$

g) $\displaystyle\lim_{x\to\infty}\frac{x+\frac{1}{x}}{x-\frac{1}{x}}$

h) $\displaystyle\lim_{x\to\infty}\frac{2^x+2^{-x}}{2^x-2^{-x}}$

i) $\displaystyle\lim_{x\to-\infty}\left(\sqrt{3x-1}-\sqrt{3x+1}\right)$

j) $\displaystyle\lim_{x\to\infty}\left(\log_3 4x-\log_3 12x\right)$

k) $\displaystyle\lim_{x\to\infty}\frac{\log_3 9x^2}{\log_3 27x}$

Practice Problems

Compute each of the following limits. Show all steps, using correct notation.

1. $\displaystyle\lim_{x\to\infty}\frac{2^{3x-1}}{5^{x-1}}$

2. $\displaystyle\lim_{x\to-\infty}\frac{2^{3x-1}}{5^{x-1}}$

3. $\displaystyle\lim_{x\to\infty}\frac{2^{2x+3}}{5^{x-1}}$

4. $\displaystyle\lim_{x\to-\infty}\frac{2^{2x+3}}{5^{x-1}}$

5. $\displaystyle\lim_{x\to\infty}\frac{2^{2x+3}}{4^{x-1}}$

6. $\displaystyle\lim_{x\to-\infty}\frac{2^{2x+3}}{4^{x-1}}$

7. $\displaystyle\lim_{x\to\infty}\frac{2^{x+3}\cdot 3^{x-1}}{7^{x-2}}$

8. $\displaystyle\lim_{x\to\infty}\frac{2^{2x+3}\cdot 3^{x-1}}{7^{x-2}}$

9. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$

10. $\displaystyle\lim_{x\to\infty}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x}}\right)$

11. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x-1}+\sqrt{x+1}}$

12. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

13. $\displaystyle\lim_{x\to\infty}x\left(\frac{1}{a}-\frac{1}{a-\dfrac{1}{x}}\right)$

14. $\displaystyle\lim_{x\to\infty}\frac{0.5^{x}+0.5^{-x}}{0.5^{x}-0.5^{-x}}$

15. $\displaystyle\lim_{x\to-\infty}\frac{0.5^{x}+0.5^{-x}}{0.5^{x}-0.5^{-x}}$

16. $\displaystyle\lim_{x\to\infty}\frac{\sin x-\cos x}{\sqrt{x^2+1}}$

17. $\displaystyle\lim_{x\to\infty}\left(2^{x+2}-2^{x}\right)$

18. $\displaystyle\lim_{x\to\infty}\frac{\log_2 4x}{\log_2 16x}$

19. $\displaystyle\lim_{x\to\infty}\left(\log_2 4x-\log_2 16x\right)$

20. $\displaystyle\lim_{x\to\infty}\left(\sqrt{2x}-\sqrt{x}\right)$

21. $\displaystyle\lim_{x\to\infty}\left(5^{x+2}-5^{x}\right)$

22. $\displaystyle\lim_{x\to\infty}\left(\frac{5^{x}}{5^{x+2}}\right)$

Sample Problems - Answers

1.) a) 0 b) 0 c) 0 d) -7 e) $-\infty$ f) 3 g) $-\infty$ h) -5 i) 0

2.) a) ∞ b) $-\infty$ c) ∞ d) ∞ 3.) a) 0 b) $\frac{1}{2}$ c) $-\infty$

4.) a) ∞ b) $\frac{3}{5}$ c) 0 d) $-1-\sqrt{2}$ e) $-\frac{1}{25}$ f) 0 g) 1 h) 1 i) undefined

j) -1 k) 2

Practice Problems - Answers

1.) ∞ 2.) 0 3.) 0 4.) ∞ 5.) 32 6.) 32 7.) 0 8.) ∞ 9.) 0

10.) 0 11.) ∞ 12.) undefined 13.) $-\frac{1}{a^2}$ 14.) -1 15.) 1 16.) 0 17.) ∞ 18.) 1

19.) -2 20.) ∞ 21.) ∞ 22.) $\frac{1}{25}$

Sample Problems - Solutions

1. Compute each of the following limits.

a) $\displaystyle\lim_{x\to\infty}\frac{1}{x}$

Solution: This is a very important limit. Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. The reciprocal of a very large positive number is a very small positive number. This limit is 0.

b) $\displaystyle\lim_{x\to-\infty}\frac{1}{x}$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. The reciprocal of a very large negative number is a very small negative number. This limit is 0.

c) $\displaystyle\lim_{x\to\infty}\frac{-5}{2x^3}$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. We divide -5 by a very large positive number. This limit is 0.

d) $\lim_{x\to-\infty}\left(\frac{-5}{2x^3}-7+\frac{8}{x}\right)$

Solution: This limit is -7 since the other two terms aproach zero as x approaches negative infinity. Using mathematical notation,

$$\lim_{x\to-\infty}\frac{-5}{2x^3}-7+\frac{8}{x}=\lim_{x\to-\infty}\frac{-5}{2x^3}+\lim_{x\to-\infty}-7+\lim_{x\to-\infty}\frac{8}{x}=0-7+0=-7$$

e) $\lim_{x\to\infty}\left(-2x^3+1-\frac{5}{x}+\frac{12}{x^4}\right)$

Solution: This limit is $-\infty$ since the first term approaches negative infinity, the second term approaches 1 and the other two terms aproach zero as x approaches infinity. Using mathematical notation,

$$\lim_{x\to\infty}\left(-2x^3+1-\frac{5}{x}+\frac{12}{x^4}\right)=\lim_{x\to\infty}\left(-2x^3\right)+\lim_{x\to\infty}1+\lim_{x\to\infty}\left(-\frac{5}{x}\right)+\lim_{x\to\infty}\left(\frac{12}{x^4}\right)=-\infty+1+0+0=-\infty$$

f) $\lim_{x\to-\infty}\frac{3x-2}{x}$

Solution: This problem is similar to the previous problems after a bit of algebra. We simply divide by x and then the limit becomes familiar.

$$\lim_{x\to-\infty}\frac{3x-2}{x}=\lim_{x\to-\infty}\left(\frac{3x}{x}-\frac{2}{x}\right)=\lim_{x\to-\infty}\left(3-\frac{2}{x}\right)=3$$

g) $\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^2}$

Solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^2} &= \lim_{x\to\infty}\left(\frac{-5x^3}{x^2}+\frac{-2x}{x^2}+\frac{4}{x^2}\right)=\lim_{x\to\infty}\left(-5x-\frac{2}{x}+\frac{4}{x^2}\right)\\ &= \lim_{x\to\infty}\left(-5x\right)+\lim_{x\to\infty}\left(-\frac{2}{x}\right)+\lim_{x\to\infty}\left(\frac{4}{x^2}\right)=-\infty+0+0=-\infty\end{aligned}$$

h) $\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^3}$

Solution:

$$\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^3}=\lim_{x\to\infty}\left(\frac{-5x^3}{x^3}+\frac{-2x}{x^3}+\frac{4}{x^3}\right)=\lim_{x\to\infty}\left(-5-\frac{2}{x^2}+\frac{4}{x^3}\right)=-5$$

i) $\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^4}$

Solution:

$$\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^4}=\lim_{x\to\infty}\left(\frac{-5x^3}{x^4}+\frac{-2x}{x^4}+\frac{4}{x^4}\right)=\lim_{x\to\infty}\left(\frac{-5}{x}-\frac{2}{x^3}+\frac{4}{x^4}\right)=0$$

2. Compute each of the following limits.

a) $\lim\limits_{x\to-\infty}\left(-2x^5-8x^4+7x^3-10\right)$

Solution: The first term, $-2x^5$ approaches infinity and the second term, $-8x^4$ approaches negative infinity. This does not give us enough information about the entire polynomial. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. In this case, factoring out the first term does the trick.

In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term. Recall that the leading term is the highest degree term.

$$\lim_{x\to-\infty}\left(-2x^5-8x^4+7x^3-10\right)=\lim_{x\to-\infty}\left(-2x^5\right)$$

Here is the computation showing why this is true. We first factor out the entire leading term.

$$\begin{aligned}\lim_{x\to-\infty}\left(-2x^5-8x^4+7x^3-10\right) &= \lim_{x\to-\infty}\left(-2x^5\right)\left(1+\frac{4}{x}-\frac{7}{2x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to-\infty}\left(-2x^5\right)\cdot\lim_{x\to-\infty}\left(1+\frac{4}{x}-\frac{7}{2x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to-\infty}\left(-2x^5\right)\cdot 1=\lim_{x\to-\infty}\left(-2x^5\right)\end{aligned}$$

We can now easily determine that this limit is ∞.

b) $\lim\limits_{x\to\infty}\left(-2x^5-8x^4+7x^3-10\right)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.** Recall that the leading term is the highest degree term.

$$\lim_{x\to\infty}\left(-2x^5-8x^4+7x^3-10\right)=\lim_{x\to\infty}\left(-2x^5\right)$$

Here is the computation showing why this is true. We first factor out the entire leading term.

$$\begin{aligned}\lim_{x\to\infty}\left(-2x^5-8x^4+7x^3-10\right) &= \lim_{x\to\infty}\left(-2x^5\right)\left(1+\frac{4}{x}-\frac{7}{2x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to\infty}\left(-2x^5\right)\cdot\lim_{x\to\infty}\left(1+\frac{4}{x}-\frac{7}{2x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to\infty}\left(-2x^5\right)\cdot 1=\lim_{x\to\infty}\left(-2x^5\right)\end{aligned}$$

We can now easily determine that this limit is $-\infty$.

c) $\lim\limits_{x\to-\infty}\left(-2x^5+8x^6\right)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.**

$$\begin{aligned}\lim_{x\to-\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to-\infty}8x^6=\infty \quad \text{because}\\ \lim_{x\to-\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to-\infty}\left(8x^6-2x^5\right)=\lim_{x\to-\infty}\left(8x^6\right)\left(1-\frac{1}{4x}\right)=\lim_{x\to-\infty}\left(8x^6\right)\cdot\lim_{x\to-\infty}\left(1-\frac{1}{4x}\right)\\ &= \lim_{x\to-\infty}\left(8x^6\right)\cdot 1=\lim_{x\to-\infty}8x^6\end{aligned}$$

We can now easily determine that this limit is ∞.

d) $\lim\limits_{x\to\infty}\left(-2x^5+8x^6\right)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.**

$$\begin{aligned}\lim_{x\to\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to\infty} 8x^6=\infty \quad \text{because}\\ \lim_{x\to\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to\infty}\left(8x^6-2x^5\right)=\lim_{x\to\infty}\left(8x^6\right)\left(1-\frac{1}{4x}\right)=\lim_{x\to\infty}\left(8x^6\right)\cdot\lim_{x\to\infty}\left(1-\frac{1}{4x}\right)\\ &= \lim_{x\to\infty}\left(8x^6\right)\cdot 1=\infty\end{aligned}$$

We can now easily determine that this limit is ∞.

3. Compute each of the following limits.

a) $\lim\limits_{x\to-\infty}\dfrac{x+x^2-6}{6x+5x^2+2x^3}=0$

Solution: The numerator approaches infinity and the denominator approaches negative infinity. This does not give us enough information about the quotient. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. Let us rearrange the polynomials in the rational function given. Then we will factor out the leading term in the numerator and denominator.

$$\lim_{x\to-\infty}\frac{x^2+x-6}{2x^3+5x^2+6x}=\lim_{x\to-\infty}\frac{x^2\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{2x^3\left(1+\dfrac{5}{2x}+\dfrac{3}{x^2}\right)}$$

We now express the limit of the product as the product of two limits

$$\lim_{x\to-\infty}\frac{x^2\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{2x^3\left(1+\dfrac{5}{2x}+\dfrac{3}{x^2}\right)}=\lim_{x\to-\infty}\frac{x^2}{2x^3}\cdot\lim_{x\to-\infty}\frac{\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{\left(1+\dfrac{5}{2x}+\dfrac{3}{x^2}\right)}$$

The first expression can be simplified and thus has a limit we can easily determine its limit. The second expression, although looks unfriendly, is always going to approach 1.

$$\lim_{x\to-\infty}\frac{x^2}{2x^3}\cdot\lim_{x\to-\infty}\frac{\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{\left(1+\dfrac{5}{2x}+\dfrac{3}{x^2}\right)}=\lim_{x\to-\infty}\frac{1}{2x}\cdot 1=0\cdot 1=0$$

The entire computation should look like this:

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^2+x-6}{2x^3+5x^2+6x} &= \lim_{x\to-\infty}\frac{x^2\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{2x^3\left(1+\dfrac{5}{2x}+\dfrac{3}{x^2}\right)}=\lim_{x\to-\infty}\frac{x^2}{2x^3}\cdot\lim_{x\to-\infty}\left(\frac{1+\dfrac{1}{x}-\dfrac{6}{x^2}}{1+\dfrac{5}{2x}+\dfrac{3}{x^2}}\right)\\ &= \lim_{x\to-\infty}\frac{1}{2x}\cdot 1=0\cdot 1=0\end{aligned}$$

b) $\displaystyle\lim_{x\to\infty}\frac{x^2+9}{5x+2x^2-3}=\frac{1}{2}$

Solution: Both numerator and denominator approach infinity. This does not give us enough information about the quotient. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. Let us rearrange the polynomials in the rational function given. Then we will factor out the leading term in the numerator and denominator.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2+9}{2x^2+5x-3} &= \lim_{x\to\infty}\frac{x^2\left(1+\dfrac{9}{x^2}\right)}{2x^2\left(1+\dfrac{5}{2x}-\dfrac{3}{2x^2}\right)}=\lim_{x\to\infty}\frac{x^2}{2x^2}\cdot\lim_{x\to\infty}\frac{1+\dfrac{9}{x^2}}{1+\dfrac{5}{2x}-\dfrac{3}{2x^2}}\\ &= \lim_{x\to\infty}\frac{1}{2}\cdot\lim_{x\to\infty}\frac{1+\dfrac{9}{x^2}}{1+\dfrac{5}{2x}-\dfrac{3}{2x^2}}=\frac{1}{2}\cdot 1=\frac{1}{2}\end{aligned}$$

c) $\displaystyle\lim_{x\to-\infty}\frac{x^3-9x+1}{3x^2-2x-15}=-\infty$

Solution:

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^3-9x+1}{3x^2-2x-15} &= \lim_{x\to-\infty}\frac{x^3\left(1-\dfrac{9}{x^2}+\dfrac{1}{x^3}\right)}{3x^2\left(1-\dfrac{2}{3x}-\dfrac{5}{x^2}\right)}=\lim_{x\to-\infty}\frac{x^3}{3x^2}\cdot\lim_{x\to-\infty}\frac{1-\dfrac{9}{x^2}+\dfrac{1}{x^3}}{\left(1-\dfrac{2}{3x}-\dfrac{5}{x^2}\right)}\\ &= \left(\lim_{x\to-\infty}\frac{x^3}{3x^2}\right)\cdot 1=\left(\lim_{x\to-\infty}\frac{x}{3}\right)\cdot 1=-\infty\cdot 1=-\infty\end{aligned}$$

4. (More indeterminates) Compute each of the following limits. Show all steps, using correct notation.

a) $\displaystyle\lim_{x\to\infty}\left(3^{x+1}-3^x\right)$

Solution: This is an "infinity minus infinity" type of an indeterminate. We need to first transform this expression until it is no longer an indeterminate.

$$\lim_{x\to\infty}\left(3^{x+1}-3^x\right)=\lim_{x\to\infty}\left(3^x\cdot 3-3^x\right)=\lim_{x\to\infty}\left(3\cdot 3^x-3^x\right)=\lim_{x\to\infty}\left(2\cdot 3^x\right)=2\lim_{x\to\infty}3^x=\infty$$

b) $\displaystyle\lim_{x\to\infty}\frac{3\sqrt{x}+2}{5\sqrt{x}+1}$

Solution: Since $\displaystyle\lim_{x\to\infty}\sqrt{x}=\infty$, clearly $\displaystyle\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0$. We will use this fact; we factor out $\sqrt{x}$ from both numerator and denominator.

$$\lim_{x\to\infty}\frac{3\sqrt{x}+2}{5\sqrt{x}+1}=\lim_{x\to\infty}\frac{\sqrt{x}\left(3+\dfrac{2}{\sqrt{x}}\right)}{\sqrt{x}\left(5+\dfrac{1}{\sqrt{x}}\right)}=\lim_{x\to\infty}\frac{3+\dfrac{2}{\sqrt{x}}}{5+\dfrac{1}{\sqrt{x}}}=\frac{3}{5}$$

c) $\displaystyle\lim_{x\to\infty}\left(\sqrt{2x-1}-\sqrt{2x}\right)$

Solution: We will transform this expression by multiplying it by 1, written as a fraction with numerator and denominator both being the conjugate of the expression.

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{2x-1}-\sqrt{2x}\right) &= \lim_{x\to\infty}\frac{\sqrt{2x-1}-\sqrt{2x}}{1}\cdot\frac{\sqrt{2x-1}+\sqrt{2x}}{\sqrt{2x-1}+\sqrt{2x}}\\ &= \lim_{x\to\infty}\frac{(2x-1)-(2x)}{\sqrt{2x-1}+\sqrt{2x}}=\lim_{x\to\infty}\frac{-1}{\sqrt{2x-1}+\sqrt{2x}}=0\end{aligned}$$

d) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

Solution: We factor out $\sqrt{x}$ from both numerator and denominator.

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}} &= \lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}\left(\dfrac{\sqrt{x+1}}{\sqrt{x}}-\sqrt{2}\right)}=\lim_{x\to\infty}\frac{1}{\sqrt{\dfrac{x+1}{x}}-\sqrt{2}}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{1+\dfrac{1}{x}}-\sqrt{2}}=\frac{1}{1-\sqrt{2}}=\frac{1}{1-\sqrt{2}}\cdot\frac{1+\sqrt{2}}{1+\sqrt{2}}=\frac{1+\sqrt{2}}{-1}\\ &= -1-\sqrt{2}\end{aligned}$$

e) $\displaystyle\lim_{x\to\infty}x\left(\frac{1}{5}-\frac{1}{5-\dfrac{1}{x}}\right)$

Solution: We just need to simplify the complex fraction. As it turns out, this problem boils down to a type we have already seen.

$$\begin{aligned}x\left(\frac{1}{5}-\frac{1}{5-\dfrac{1}{x}}\right) &= x\left(\frac{1}{5}-\frac{1}{\dfrac{5x-1}{x}}\right)=x\left(\frac{1}{5}-\frac{x}{5x-1}\right)=x\left(\frac{(5x-1)-5x}{5(5x-1)}\right)\\ &= x\left(\frac{5x-1-5x}{5(5x-1)}\right)=x\frac{-1}{25x-5}=\frac{-x}{25x-5}\end{aligned}$$

Thus

$$\lim_{x\to\infty}x\left(\frac{1}{5}-\frac{1}{5-\dfrac{1}{x}}\right)=\lim_{x\to\infty}\frac{-x}{25x-5}=\lim_{x\to\infty}\frac{x(-1)}{x\left(25-\dfrac{5}{x}\right)}=-\frac{1}{25}$$

f) $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}$

Solution: This problem can be solved by the sandwich principle. Consider the limits $\displaystyle\lim_{x\to\infty}\frac{1}{x}$ and $\displaystyle\lim_{x\to\infty}\left(-\frac{1}{x}\right)$. These limits are both zero. Furthermore, since

$$\begin{aligned}-1 &\le \cos x\le 1 &&\text{for all } x\text{, we also have}\\ -\frac{1}{x} &\le \frac{\cos x}{x}\le\frac{1}{x} &&\text{for all positive } x\end{aligned}$$

Our function $f(x)=\dfrac{\cos x}{x}$ is 'locked' between $g(x)=\dfrac{1}{x}$ and $h(x)=-\dfrac{1}{x}$. Since these both approach zero, so must the function $f(x)=\dfrac{\cos x}{x}$. Thus $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}=0$.

g) $\displaystyle\lim_{x\to\infty}\frac{x+\frac{1}{x}}{x-\frac{1}{x}}$

Solution: We will factor out x from both numerator and denominator, and use the fact that $\displaystyle\lim_{x\to\infty}\frac{1}{x^k}=0$ for all positive integers k.

$$\lim_{x\to\infty}\frac{x+\frac{1}{x}}{x-\frac{1}{x}}=\lim_{x\to\infty}\frac{x\left(1+\frac{1}{x^2}\right)}{x\left(1-\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}=1$$

h) $\displaystyle\lim_{x\to\infty}\frac{2^x+2^{-x}}{2^x-2^{-x}}$

Solution: First, $\displaystyle\lim_{x\to\infty}2^x=\infty$ (and so 2^x is large) and $\displaystyle\lim_{x\to\infty}2^{-x}=0$ (and so 2^{-x} is small). With that in mind, this limit is similar to $\displaystyle\lim_{x\to\infty}\frac{x+\frac{1}{x}}{x-\frac{1}{x}}$. The solution also will be similar. We will factor out 2^x from both numerator and denominator.

$$\lim_{x\to\infty}\frac{2^x+2^{-x}}{2^x-2^{-x}}=\lim_{x\to\infty}\frac{2^x+\frac{1}{2^x}}{2^x-\frac{1}{2^x}}=\lim_{x\to\infty}\frac{2^x\left(1+\frac{1}{(2^x)^2}\right)}{2^x\left(1-\frac{1}{(2^x)^2}\right)}=\lim_{x\to\infty}\frac{1+\frac{1}{2^{2x}}}{1-\frac{1}{2^{2x}}}=1$$

i) $\displaystyle\lim_{x\to-\infty}\left(\sqrt{3x-1}-\sqrt{3x+1}\right)$

Solution: When $x\to-\infty$, then we may assume it is negative. Then the expressions under the square root are negative and the function is not defined. Thus, there is no limit at negative infinity. The answer is: undefined.

j) $\displaystyle\lim_{x\to\infty}(\log_3 4x-\log_3 12x)$

Solution: This is an infinity minus infinity type of an indeterminate. We will use properties of lgarithms to bring it to a form that is no longer an indeterminate. Recall the rule $\log_3 a-\log_3 b=\log_3\left(\frac{a}{b}\right)$

$$\lim_{x\to\infty}(\log_3 4x-\log_3 12x)=\lim_{x\to\infty}\left(\log_3\frac{4x}{12x}\right)=\lim_{x\to\infty}\left(\log_3\frac{4}{12}\right)=\lim_{x\to\infty}\left(\log_3\frac{1}{3}\right)=\lim_{x\to\infty}(-1)=-1$$

k) $\displaystyle\lim_{x\to\infty}\frac{\log_3 9x^2}{\log_3 27x}$

Solution: This is an infinity divided by infinity type of an indeterminate. We will use properties of lgarithms to bring it to a form that is no longer an indeterminate. Recall the rules $\log_3 a+\log_3 b=\log_3 ab$ and $\log_3\left(a^b\right)=b\log_3 a$.

$$\lim_{x\to\infty}\frac{\log_3 9x^2}{\log_3 27x}=\lim_{x\to\infty}\frac{\log_3 9+\log_3 x^2}{\log_3 27+\log_3 x}=\lim_{x\to\infty}\frac{2+2\log_3 x}{3+\log_3 x}$$

Since $\displaystyle\lim_{x\to\infty}\log_3 x=\infty$, this is still an indeterminate of an infinity divided by infinity type. However, it is very similar to $\displaystyle\lim_{x\to\infty}\frac{2+2x}{3+x}$ and so we will transform it using the same technique. We will factor out $\log_3 x$ from both numerator and denominator and use the fact that $\lim\limits_{x\to\infty} \dfrac{1}{\log_3 x} = 0$.

$$\lim_{x\to\infty} \frac{2+2\log_3 x}{3+\log_3 x} = \lim_{x\to\infty} \frac{\log_3 x\left(\dfrac{2}{\log_3 x}+2\right)}{\log_3 x\left(\dfrac{3}{\log_3 x}+1\right)} = \frac{2}{1} = 2$$