Sample Problems

1. We would like to construct an open box with a square base. If the volume of this box is to be $60\,\text{ft}^2$, what dimensions would guarantee that we are using the least amount of material to construct this box?

2. We would like to construct an open box with a square base. The box should have a volume of $100\,\text{ft}^3$. The material for the sides costs 3 cents per square feet, and the material for the bottom costs 5 cents per square feet. What is the lowest cost for which such a box can be produced?

3. We want to construct a cylindrical soda can with volume $100\,\text{cm}^3$. If the material for the side costs 2 cents per cm^2, and the material for the top and bottom costs 5 cents per cm^2, what dimensions would guarantee a minimal cost of producing such a can? What would be the minimal cost?

Solutions

1. We would like to construct an open box with a square base. If the volume of this box is to be 60 ft^2, what dimensions would guarantee that we are using the least amount of material to construct this box?

   $V = x^2 h \quad \text{so } h = \dfrac{V}{x^2} \qquad A = x^2 + 4xh = x^2 + 4x\left(\dfrac{V}{x^2}\right) = x^2 + 4\dfrac{V}{x} = x^2 + 4Vx^{-1}$

   $A'(x) = \dfrac{d}{dx}\left(x^2 + 4Vx^{-1}\right) = 2x - 4\dfrac{V}{x^2} \qquad 2x = 4\dfrac{V}{x^2} \qquad x^3 = 2V \qquad x = \sqrt[3]{2V}$

   how do we know this is a mimium? $\quad A''(x) = \dfrac{d}{dx}\left(2x - 4\dfrac{V}{x^2}\right) = \dfrac{8V}{x^3} + 2$ which is positive for all $x > 0$

   $A'' > 0$ indicates minimum

2. We would like to construct an open box with a square base. The box should have a volume of 100 ft^3. The material for the sides costs 3 cents per square feet, and the material for the bottom costs 5 cents per square feet. What is the lowest cost for which such a box can be produced?

   \$3.65 with base $\sqrt[3]{120}$ ft by $\sqrt[3]{120}$ ft

3. We want to construct a cylindrical soda can with volume 100 cm^3. If the material for the side costs 2 cents per cm^2, and the material for the top and bottom costs 5 cents per cm^2, what dimensions would guarantee a minimal cost of producing such a can? What would be the minimal cost?

   $V = \pi r^2 h \qquad h = \dfrac{V}{\pi r^2}$

   $C = 2\pi r^2(5) + 2\pi r h(2) = 10\pi r^2 + 4\pi h r = 10\pi r^2 + 4\pi r\left(\dfrac{V}{\pi r^2}\right) = 10\pi r^2 + 4\dfrac{V}{r}$

   $C(r) = 10\pi r^2 + 4\dfrac{V}{r} \qquad C'(r) = 20\pi r - 4\dfrac{V}{r^2} \qquad 20\pi r - 4\dfrac{V}{r^2} = 0 \qquad 5\pi r = \dfrac{V}{r^2} \qquad r^3 = \dfrac{V}{5\pi}$

   $r = \sqrt[3]{\dfrac{V}{5\pi}}$

   $$h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\sqrt[3]{\frac{V}{5\pi}}\right)^2} = \frac{V\cdot\sqrt[3]{\frac{V}{5\pi}}}{\pi\left(\sqrt[3]{\frac{V}{5\pi}}\right)^2\cdot\sqrt[3]{\frac{V}{5\pi}}} = \frac{V\cdot\sqrt[3]{\frac{V}{5\pi}}}{\pi\frac{V}{5\pi}} = \frac{\sqrt[3]{\frac{V}{5\pi}}}{\frac{1}{5}} = 5\sqrt[3]{\frac{V}{5\pi}} = 5r$$

   $r = \sqrt[3]{\dfrac{20}{\pi}}$ cm $\qquad h = 5\sqrt[3]{\dfrac{20}{\pi}}$ cm $\qquad$ cost: 323.74 cents

