Sample Problems

1. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.

2. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?

3. A company determines that the if n is the number of items produced, they can all be sold at a price of $p(n) = \sqrt{1200 - 0.2n}$ What is the greatest revenue possible?

4. A company has \$120 000 to spend on the development and promotion of a new product. The company estimates that if x is spent on the development and y is spent on promotion, then approximately $\dfrac{x^{1/2}y^{3/2}}{400\,000}$ items of new product will be sold. Based on this estimate, what is the maximum number of products that the company can sell?

5. Consider the function $g(x) = \dfrac{-2x}{(x^2+1)^2}$.

   a) Find all relative extrema of g.
   b) Find all absolute extrema of g.
   c) Find all values of c for which the function $f(x) = \dfrac{1}{x^2+1} + cx$ is increasing on its entire domain.

6. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

7. An underground telephone cable is to be laid between two boat docks on opposite banks of a straight river. One boathouse is 600 meters downstream from the other. The river is 200 meters wide. If the cost of laying the cable is \$50 per meter under water and \$30 per meter on land, how should the cable to be laid to minimize cost?
   a) Find the lowest cost possible.
   b) Use the second derivative test to prove that we found a relative minimum in part a).
   c) Prove that the relative minimum we found in part a) is also an absolue minimum on the domain $[0, 600]$.
   d) Prove that if we define C on the set of all real numbers, then the minimum we found in part a) is still an absolute minimum.

8. The location function of an object is $s(t) = \dfrac{120}{1+2e^{-t}}$ where t is time, measured in seconds. Where is the object when it is moving with the greatest speed? When is that greatest speed achieved?

Sample Problems - Answers

1. 40
2. no,the lowest possible cost is \$300 when the box is to be 5 m by 5 m by 10 m
3. \$80 000
4. 11 691
5. a) $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ relative maximum and a relative minimum $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ b) see solutions c) $c \geq \frac{3\sqrt{3}}{8}$
6. $r = \sqrt[3]{\frac{500}{\pi}} \simeq 5.419\,26$ and $h = 2\sqrt[3]{\frac{500}{\pi}} = 2r \simeq 10.838\,521\,402\,785\,8$
7. a) \$26 000 when the cable is laid 450 meters on the ground b) see solutions c) see solutions d) see solutions
8. $s = 60$ when $t = \ln 2$

Sample Problems - Solutions

1. Let a and b be positive numbers such that $ab = 10$. Find the lowest value of $a^2 + 4b^2$.

   Solution: We solve for a in terms of b: $a = \dfrac{10}{b}$. Then the expression $a^2 + 4b^2$ becomes

$$P(b) = \left(\frac{10}{b}\right)^2 + 4b^2 = 4b^2 + \frac{100}{b^2} = 4b^2 + 100b^{-2}$$

   We differentiate this:

$$\begin{aligned} P'(b) &= 8b + 100(-2)b^{-3} = 8b - \frac{200}{b^3} = \frac{8b^4 - 200}{b^3} \\ &= \frac{8\left(b^4 - 25\right)}{b^3} = \frac{8\left(b^2+5\right)\left(b^2-5\right)}{b^3} = \frac{8\left(b^2+5\right)\left(b+\sqrt{5}\right)\left(b-\sqrt{5}\right)}{b^3} \end{aligned}$$

   The critical numbers for P are $-\sqrt{5}$, 0, and $\sqrt{5}$. All relative maximums or minimums will be here. We can figure out when P' is positive and negative by sorting out the signs of each factor in the numerator and denominator.

	$b < -\sqrt{5}$	$-\sqrt{5} < b < 0$	$0 < b < \sqrt{5}$	$b > \sqrt{5}$
$\left(b^2+5\right)$	$+$	$+$	$+$	$+$
$\left(b+\sqrt{5}\right)$	$-$	$+$	$+$	$+$
$\left(b-\sqrt{5}\right)$	$-$	$-$	$-$	$+$
b^3	$-$	$-$	$+$	$+$
P'	$-$	$+$	$-$	$+$

   Based on the signs of P' only, P has a relative minimum at $b = -\sqrt{5}$ and $\sqrt{5}$ and a relative maximum at 0. However, the function does not have a relative maximum at zero. Looking at the formula for the original function, $P(b) = 4b^2 + \dfrac{100}{b^2}$, we see that there is a vertical asymptote and the graph shoots up toward plus infinity on both sides of the asymptote. Not to mention tha fact that a and b must both be positive. Since b must be positive, we may consider P on the domain $(0, \infty)$. On this domain, P is continuous and differentiable everywhere, is decreasing on $\left(0, \sqrt{5}\right)$ and increasing on $\left(\sqrt{5}, \infty\right)$ and so P has an absolute minimum at $b = \sqrt{5}$.

   If $b = \sqrt{5}$, then

$$P\left(\sqrt{5}\right) = \left(\frac{10}{\sqrt{5}}\right)^2 + 4\left(\sqrt{5}\right)^2 = \frac{100}{5} + 4 \cdot 5 = 20 + 20 = 40$$

   Thus the smallest possible value of $a^2 + 4b^2$ is 40.

2. A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Can the box be constructed for less than \$300?

   Solution: Let x denote the side of the square base, and h denote the height of the box. Then $V = hx^2$ gives us

$$\begin{aligned} hx^2 &= 250 \\ h &= \frac{250}{x^2} \end{aligned}$$

We now set up the cost function, $C(x)$. The top and bottom each cost \$2 per square meter, and have area x^2. The four sides each have area $xh = x\left(\frac{250}{x^2}\right) = \frac{250}{x}$ and cost \$1 per square meter. Thus

$$C(x) = 2 \cdot 2 \cdot x^2 + 4 \cdot 1 \cdot \frac{250}{x} = 4x^2 + \frac{1000}{x} = 4x^2 + 1000x^{-1}$$

We are looking for the maximum of $C(x)$. We will differentiate C first.

$$\begin{aligned} C'(x) &= 8x + 1000(-1)x^{-2} = 8x - \frac{1000}{x^2} = \frac{8x^3 - 1000}{x^2} = \frac{8(x^3 - 125)}{x^2} \\ &= \frac{8(x-5)(x^2 + 5x + 25)}{x^2} \end{aligned}$$

The last form shows that C' has only one zero, at $x = 5$. Since both x^2 and $x^2 + 5x + 25$ are positive for all values of x, C' will change sign from negative to positive at $x = 5$, indicating a minimum of C. Thus, the lowest possible cost will be associated with $x = 5$. The actual cost is then

$$C(5) = 4 \cdot 5^2 + \frac{1000}{5} = 300$$

Thus, we can not construct this box for less than \$300.

3. A company determines that the if n is the number of items produced, they can all be sold at a price of $p(n) = \sqrt{1200 - 0.2n}$ What is the greatest revenue possible?
Solution: If we sell all n products, the revenue is $R(n) = n\sqrt{1200 - 0.2n}$ Since the price should be positive, the domain of this function is determined by $n > 0$ and $1200 - 0.2n > 0$. We solve these inequalities and obtain the domain. $(0, 6000)$. We will find the maximum of this function by finding the zeroes of the derivative.

$$\begin{aligned} R'(n) &= \sqrt{1200 - 0.2n} + n\frac{1}{2\sqrt{1200 - 0.2n}}(-0.2) = \sqrt{1200 - 0.2n} - \frac{0.1n}{\sqrt{1200 - 0.2n}} \\ &= \frac{1200 - 0.2n}{\sqrt{1200 - 0.2n}} - \frac{0.1n}{\sqrt{1200 - 0.2n}} = \frac{1200 - 0.2n - 0.1n}{\sqrt{1200 - 0.2n}} = \frac{-0.3n + 1200}{\sqrt{1200 - 0.2n}} \\ &= \frac{-0.3(n - 4000)}{\sqrt{1200 - 0.2n}} \end{aligned}$$

The derivative has only one zero, at $n = 4000$. The denominator is positive for all n with $0 < n < 6000$ and the numerator is positive before 4000 and negative after 4000. Consequently, $R(n)$ has a relative and absolute maximum at $n = 4000$. The maximal possible revenue is then $R(4000)$.

$$R(4000) = 4000\sqrt{1200 - 0.2(4000)} = 4000\sqrt{1200 - 800} = 4000\sqrt{400} = 4000(20) = 80\,000$$

4. A company has \$120 000 to spend on the development and promotion of a new product. The company estimates that if x is spent on the development and y is spent on promotion, then approximately $\frac{x^{1/2}y^{3/2}}{400\,000}$ items of new product will be sold. Based on this estimate, what is the maximum number of products that the company can sell?
Solution: Since $y = 120\,000 - x$, we can write the number sold as a function of x.

$$N(x) = \frac{1}{400\,000}x^{1/2}(120\,000 - x)^{3/2}$$

The domain is clearly $[0, 120\,000]$. We differentiate N.

$$\begin{aligned} N'(x) &= \frac{1}{400\,000}\left[\frac{1}{2}x^{-1/2}(120\,000 - x)^{3/2} + x^{1/2}\left(\frac{3}{2}\right)(120\,000 - x)^{1/2}(-1)\right] \\ &= \frac{1}{400\,000}\left(\frac{1}{2}\right)\left[\frac{(120\,000 - x)^{3/2}}{\sqrt{x}} - 3\sqrt{x}(120\,000 - x)^{1/2}\right] \end{aligned}$$

$$\begin{aligned}
N'(x) &= \frac{1}{800\,000}\left(\frac{\sqrt{120\,000-x}\,(120\,000-x)}{\sqrt{x}} - 3\sqrt{x}\sqrt{120\,000-x}\right) && \text{factor out } \sqrt{120\,000-x} \\
&= \frac{\sqrt{120\,000-x}}{800\,000}\left(\frac{(120\,000-x)}{\sqrt{x}} - 3\sqrt{x}\right) && \text{bring difference to common denominator} \\
&= \frac{\sqrt{120\,000-x}}{800\,000}\left(\frac{(120\,000-x)}{\sqrt{x}} - 3\frac{x}{\sqrt{x}}\right) = \frac{\sqrt{120\,000-x}}{800\,000}\left(\frac{120\,000-x-3x}{\sqrt{x}}\right) \\
&= \frac{\sqrt{120\,000-x}}{800\,000}\left(\frac{120\,000-4x}{\sqrt{x}}\right) = \frac{-4\sqrt{120\,000-x}\,(x-30\,000)}{800\,000\sqrt{x}} \\
&= \frac{\sqrt{120\,000-x}}{800\,000\sqrt{x}}\left[-4\,(x-30\,000)\right]
\end{aligned}$$

$$N'(x) = \frac{\sqrt{120\,000-x}}{800\,000\sqrt{x}}\quad(-4)\,(x-30\,000)$$

The first factor, $\dfrac{\sqrt{120\,000-x}}{800\,000\sqrt{x}}$ is positive for all x in the domain, except for the endpoints. The second factor, $-4\,(x-30\,000)$ is positive before $30\,000$, zero at $x=30\,000$, and negative after $30\,000$, indicating an absolute maximum at $x=30\,000$. Then $y=120\,000-30\,000=90\,000$ and the maximal number sold is then

$$N(30\,000) = \frac{x^{1/2}y^{3/2}}{400\,000} = \frac{30\,000^{1/2}90\,000^{3/2}}{400\,000} \approx 11691.342\,95$$

The integer nearest to this number is $11\,691$.

5. Consider the function $g(x) = \dfrac{-2x}{(x^2+1)^2}$.

a) Find all relative extrema of g.

Solution: We differentiate g using the quotient rule.

$$\begin{aligned}
g'(x) &= \frac{-2\left(x^2+1\right)^2 - (-2x)\,2\left(x^2+1\right)(2x)}{\left(x^2+1\right)^4} && \text{simplify by } x^2+1 \\
&= \frac{-2\left(x^2+1\right) - (-2x)\,2\,(2x)}{\left(x^2+1\right)^3} = \frac{-2x^2-2+8x^2}{\left(x^2+1\right)^3} = \frac{6x^2-2}{\left(x^2+1\right)^3}
\end{aligned}$$

We now factor the numerator to see when g' is positive and negative.

$$g'(x) = \frac{6x^2-2}{\left(x^2+1\right)^3} = \frac{6\left(x^2-\frac{1}{3}\right)}{\left(x^2+1\right)^3} = \frac{6\left(x+\frac{1}{\sqrt{3}}\right)\left(x-\frac{1}{\sqrt{3}}\right)}{\left(x^2+1\right)^3}$$

The denominator is always positive, the numerator is a quadratic expression with a positive leading coefficient, and so g' is positive on $\left(-\infty,-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}},\infty\right)$ and negative on $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$. Consequently, g is increasing on $\left(-\infty,-\frac{1}{\sqrt{3}}\right)$, decreasing on $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, and increasing on $\left(\frac{1}{\sqrt{3}},\infty\right)$. That means that g has a relative maximum at $x=-\frac{1}{\sqrt{3}}$ and a relative minimum at $x=\frac{1}{\sqrt{3}}$. We compute the function values at $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

$$g\left(-\frac{1}{\sqrt{3}}\right) = \frac{-2\left(-\frac{1}{\sqrt{3}}\right)}{\left(\left(-\frac{1}{\sqrt{3}}\right)^2+1\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\left(\frac{1}{3}+1\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\left(\frac{4}{3}\right)^2} = \frac{2}{\sqrt{3}}\cdot\frac{9}{16} = \frac{3\sqrt{3}}{8}$$

This is a good time to notice that g is an odd function

$$g(-x) = \frac{-2(-x)}{\left((-x)^2+1\right)^2} = \frac{2x}{(x^2+1)^2} = -g(x)$$

and so g has a relative maximum: $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ and a relative minimum $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$.

b) Find all absolute extrema of g.
Solution: We will show that the relateive extrema we found in part a) are in fact absolute extrema. This is not a fact that simply follows from the signs of the derivative. For all we know, a decreasing, then increasing, then decreasing function may look like an upside down cubic function that has neither absolute minumum nor absolute maximum.

Claim: $\left(-\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{8}\right)$ is an absolute maximum, i.e. for all real numbers x, $g(x) \le \frac{3\sqrt{3}}{8}$.

In part a) we determined that g is increasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

Because g is increasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$,

$$\text{for all } x \le -\frac{1}{\sqrt{3}}, \qquad g(x) \le g\left(-\frac{1}{\sqrt{3}}\right)$$

Because g is decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$$\text{for all } -\frac{1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{3}}, \qquad g(x) \le g\left(-\frac{1}{\sqrt{3}}\right)$$

We only have to prove that for all $x \ge \frac{1}{\sqrt{3}}$, $\quad g(x) \le g\left(-\frac{1}{\sqrt{3}}\right)$ is also true, but the increasing/decreasing behavior does not help here, since g is increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$. Luckily, there is a simple, elementary way to finish the proof. Recall that $g(x) = \frac{-2x}{(x^2+1)^2}$. It is easy to see that if x is positive, then $g(x)$ is negative. Thus g is negative on $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and so

$$\text{for all } x \ge \frac{1}{\sqrt{3}}, \qquad g(x) \le g\left(-\frac{1}{\sqrt{3}}\right) = \frac{3\sqrt{3}}{8}$$

Thus g has an absolute maximum at $x = -\frac{1}{\sqrt{3}}$.

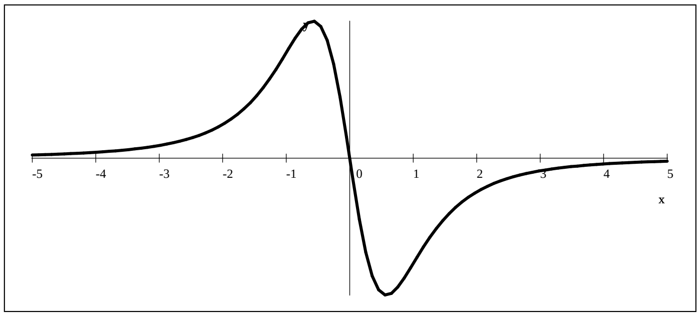
Claim: $\left(\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{8}\right)$ is an absolute minimum, i.e. for all real numbers x, $g(x) \le -\frac{3\sqrt{3}}{8}$.

proof: a very similar argument could work. g is decreasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$

and positive on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ Another way to prove this is to use the fact that g is an odd function.

$$\begin{aligned} \text{for all } x,\ g(x) &\leq \frac{3\sqrt{3}}{8} && \text{multiply by } -1 \\ \text{for all } x,\ -g(x) &\geq -\frac{3\sqrt{3}}{8} && \text{since } g \text{ is odd, } -g(x) = g(-x) \\ \text{for all } x,\ g(-x) &\geq -\frac{3\sqrt{3}}{8} \end{aligned}$$

and so g has an absolute minimum at $x = \frac{1}{\sqrt{3}}$.



c) Find all values of c for which the function $f(x) = \frac{1}{x^2+1} + cx$ is increasing on its entire domain.

Solution: $f(x)$ is always increasing if $f'(x)$ is always positive. $f'(x) = \frac{-2x}{(x^2+1)^2} + c$. We proved that the absolute minimum of $\frac{-2x}{(x^2+1)^2}$ is $-\frac{3\sqrt{3}}{8}$ and so if we set $c \geq \frac{3\sqrt{3}}{8}$, then $f'(x) = \frac{-2x}{(x^2+1)^2} + c$ will be non-negative for all x.

6. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?
Solution: Let h denote the height of the can, and r denote the radius of the base circle.

$$\pi r^2 h = 1000 \qquad h = \frac{1000}{\pi r^2}$$

The domain is $(0, \infty)$

$$\begin{aligned} S(r) &= 2\pi r h + 2\pi r^2 = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r} \\ S'(r) &= 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} \end{aligned}$$

$$\begin{aligned} 4\pi r - \frac{2000}{r^2} &= 0 \\ 4\pi r &= \frac{2000}{r^2} \\ \pi r^3 &= 500 \quad \Longrightarrow \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419\,26 \end{aligned}$$

and

$$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi\left(500^{2/3}\right)\left(\pi^{-2/3}\right)} = \frac{2 \cdot 500}{\left(500^{2/3}\right)\left(\pi^{1/3}\right)} = \frac{2 \cdot 500^{1/3}}{\left(\pi^{1/3}\right)} = 2\sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.838\,52$$

But is this an absolute minimum we found?

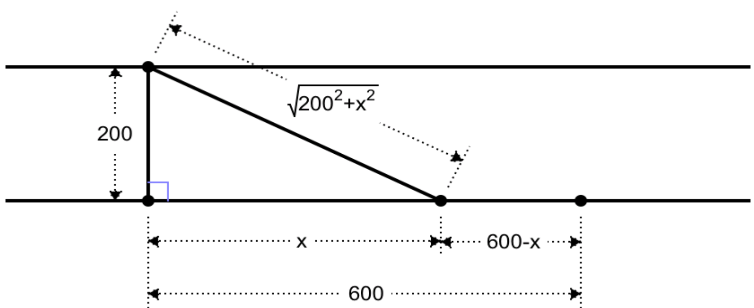
$$S''(r) = 4\pi + \frac{4000}{r^3}$$

Since S'' is positive on the entire domain (recall $r > 0$), S' is strictly increasing on its entire domain. This means that S' is negative before its only zero and positive after. This implies that S is decreasing before and increasing after, and so we indeed found the absolute minimum.

7. An underground telephone cable is to be laid between two boat docks on opposite banks of a straight river. One boathouse is 600 meters downstream from the other. The river is 200 meters wide. If the cost of laying the cable is \$50 per meter under water and \$30 per meter on land, how should the cable to be laid to minimize cost?

a) Find the lowest cost possible.

Solution: Let us denote by x - as shown on the picture below - the distance alongside the river of the part of the cable to be laid under the water.



Then the length of the cable laid under water is $\sqrt{200^2 + x^2}$ and $600 - x$ on the ground. We can now express the cost as a function of x.

$$C(x) = 50\sqrt{200^2 + x^2} + 30(600 - x) \qquad \text{on domain } [0, 600]$$

We find the minimum of C by differentiating C.

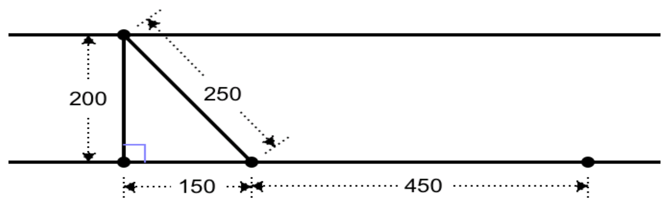
$$C'(x) = 50\frac{1}{2\sqrt{200^2 + x^2}}(2x) + 30(-1) = \frac{50x}{\sqrt{200^2 + x^2}} - 30$$

To find the extrema, we solve for the zeroes of the derivative.

$$\begin{aligned} C'(x) &= 0 \\ \frac{50x}{\sqrt{200^2 + x^2}} - 30 &= 0 \\ \frac{50x}{\sqrt{200^2 + x^2}} &= 30 \\ 50x &= 30\sqrt{200^2 + x^2} \\ 5x &= 3\sqrt{200^2 + x^2} \\ 25x^2 &= 9\left(200^2 + x^2\right) \end{aligned}$$

$$\begin{aligned} 25x^2 &= 9\left(x^2 + 40\,000\right) \\ 25x^2 &= 9x^2 + 360\,000 \\ 16x^2 &= 360\,000 \\ x^2 &= 22\,500 \\ x &= \pm 150 \end{aligned}$$

Since the domain is $[0, 600]$, the only possibility is $x = 150$. Thus the minimum cost is $C(150)$.



$$C(150) = \$50\sqrt{200^2+150^2} + \$30\,(600-150) = \$50\cdot 250 + \$30\cdot 450 = \$26\,000$$

b) Use the second derivative test to prove that we found a relative minimum in part a).
Solution: We already know that $C'(150)=0$. If $C''(150)$ is positive, then C has a relative minimum at $x=150$. If $C''(150)$ is negative, then C has a relative maximum at $x=150$. If $C''=0$, the second derivative test is inconclusive.

$$\begin{aligned}
C'(x) &= \frac{50x}{\sqrt{200^2+x^2}} - 30 \\
C''(x) &= \frac{50\sqrt{200^2+x^2} - 50x\dfrac{1}{2\sqrt{200^2+x^2}}(2x)}{200^2+x^2} = \frac{50\sqrt{200^2+x^2} - \dfrac{50x^2}{\sqrt{200^2+x^2}}}{200^2+x^2} \\
C''(x) &= \frac{50}{200^2+x^2}\left(\sqrt{200^2+x^2} - \frac{x^2}{\sqrt{200^2+x^2}}\right) \\
C''(150) &= \frac{50}{200^2+150^2}\left(\sqrt{200^2+150^2} - \frac{150^2}{\sqrt{200^2+150^2}}\right) = \frac{16}{125} > 0
\end{aligned}$$

Since its second derivative is positive at $x=150$, the function C has a relative minimum at $x=150$.

c) Prove that the relative minimum we found in part a) is also an absolue minimum on the domain $[0,600]$.
Solution: Since the function $C(x)$ is continuous on the closed interval $[0,600]$, it achieves the ansolute minimum and absolute maximum. To find these, we only need to evaluate the function at the relative extrema and the endpoints of the interval. We need to compute $C(0)$, $C(150)$, and $C(600)$. These values are $C(0)=\$28\,000$, $C(150)=\$26\,000$, and $C(600)=\$10\,000\sqrt{10}\approx \$31622.776\,602$. Thus C has an absolute minimum at $x=150$.

d) Prove that if we define C on the set of all real numbers, then the minimum we found in part a) is still an absolute minimum.
Solution: Finding absolute extrema on domains others than a closed interval is often tricky. In this case, we can prove the statement by looking at C'' more carefully.

$$\begin{aligned}
C''(x) &= \frac{50}{200^2+x^2}\left(\sqrt{200^2+x^2} - \frac{x^2}{\sqrt{200^2+x^2}}\right) = \frac{50}{200^2+x^2}\left(\frac{\left(\sqrt{200^2+x^2}\right)^2}{\sqrt{200^2+x^2}} - \frac{x^2}{\sqrt{200^2+x^2}}\right) \\
&= \frac{50}{200^2+x^2}\left(\frac{200^2+x^2-x^2}{\sqrt{200^2+x^2}}\right) = \frac{50}{200^2+x^2}\left(\frac{200^2}{\sqrt{200^2+x^2}}\right)
\end{aligned}$$

Since C'' is positive for all x, C' is increasing for all x. So the zero at $x=150$ is the only zero of C'; C' is negative on $(-\infty,150)$ and positive on $(150,\infty)$. Consequently, C is decreasing on on $(-\infty,150)$ and increasing on $(150,\infty)$, and so C has an absolute minimum at $x=150$.

8. The location function of an object is $s(t) = \dfrac{120}{1+2e^{-t}}$ where t is time, measured in seconds. Where is the object when it is moving with the greatest speed? When is that greatest speed achieved?
Solution:

$$\begin{aligned} s(t) &= \frac{120}{1+2e^{-t}} = 120\left(1+2e^{-t}\right)^{-1} \\ v(t) &= s'(t) = 120(-1)\left(1+2e^{-t}\right)^{-2}\left(2e^{-t}\right)(-1) = \frac{240e^{-t}}{\left(1+2e^{-t}\right)^2} = 240\frac{e^{-t}}{\left(1+2e^{-t}\right)^2} \end{aligned}$$

We need to find the maximum of v which means we need to differentiate again. This time we will need to use the quotient rule.

$$\begin{aligned} a(t) &= v'(t) = 240\frac{e^{-t}(-1)\left(1+2e^{-t}\right)^2 - e^{-t}(2)\left(1+2e^{-t}\right)\left(2e^{-t}\right)(-1)}{\left(1+2e^{-t}\right)^4} = \quad \text{simplify by } \left(1+2e^{-t}\right) \\ &= 240\frac{e^{-t}(-1)\left(1+2e^{-t}\right) - e^{-t}(2)\left(2e^{-t}\right)(-1)}{\left(1+2e^{-t}\right)^3} \qquad \text{factor out } e^{-t} \\ &= 240e^{-t}\frac{-\left(1+2e^{-t}\right)+4e^{-t}}{\left(1+2e^{-t}\right)^3} = 240e^{-t}\frac{-1-2e^{-t}+4e^{-t}}{\left(1+2e^{-t}\right)^3} = 240e^{-t}\frac{2e^{-t}-1}{\left(1+2e^{-t}\right)^3} \end{aligned}$$

$240e^{-t}$ and the denominator are always positive. The only way $v'(t) = 0$ is when $2e^{-t} - 1 = 0$. We solve this equation for t.

$$\begin{aligned} 2e^{-t} - 1 &= 0 & -t &= \ln\left(\frac{1}{2}\right) \\ 2e^{-t} &= 1 & t &= -\ln\left(\frac{1}{2}\right) = \ln 2 \\ e^{-t} &= \frac{1}{2} \end{aligned}$$

Thus $t = \ln 2$ is when the object is fastest. The location of the object is then

$$s(\ln 2) = \frac{120}{1+2e^{-\ln 2}} = \frac{120}{1+2\left(\frac{1}{2}\right)} = \frac{120}{2} = 60$$

