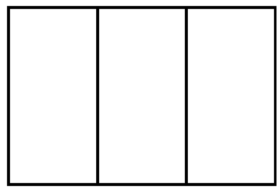
Sample Problems

1. One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.
2. We have P meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure below. What is the maximal area we can enclose this way?



3. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?
4. Prove that for any real numbers a and b, if $a+b=1$, then $a^4+b^4 \geq \frac{1}{8}$.
5. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

Sample Problems - Answers

1.) 4 and 32 2.) $\frac{P^2}{32}$

3.) 10 cm by 10 cm 4.) see solutions

5.) $r = \sqrt[3]{\frac{500}{\pi}}$ cm ≈ 5.41926 cm and $h = 2r = 2\sqrt[3]{\frac{500}{\pi}}$ cm ≈ 10.83852 cm

Sample Problems - Solutions

1. One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.
Solution: Let us denote the numbers by a and b. We have $a + b^2 = 48 \implies a = 48 - b^2$. Then the product of the two numbers is a function of b,

$$P(b) = b\left(48 - b^2\right) = -b^3 + 48b$$

Since this is a cubic function, it does not have an absolute maximum, only if its domain is restricted. The domain is

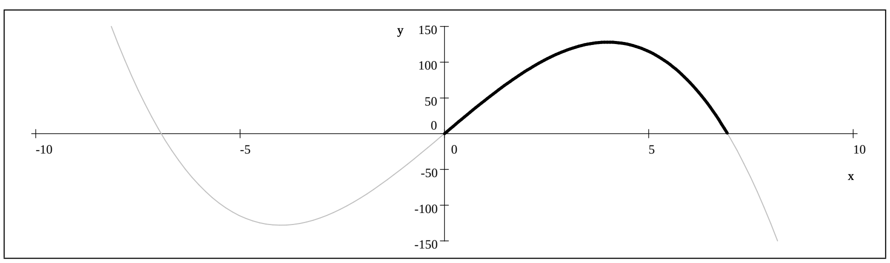
$$b \text{ is positive } \implies b > 0 \quad \text{and}$$

$$a \text{ is positive } \implies 48 - b^2 > 0 \text{ we solve this for } b \text{ and get}$$

$$\begin{aligned} 48 - b^2 &> 0 \\ 48 &> b^2 \\ -\sqrt{48} &< b < \sqrt{48} \end{aligned}$$

which means that the domain of P is $\left(0, \sqrt{48}\right)$. (This is the set of all numbers for which both a and b are positive.) We can sketch P; it is a cubic polynomial with a negative leading coefficient, and its zeroes are at $x = 0, -\sqrt{48}, \sqrt{48}$ as its factored form is

$$P(b) = -b\left(b^2 - 48\right) = -b\left(b + \sqrt{48}\right)\left(b - \sqrt{48}\right)$$

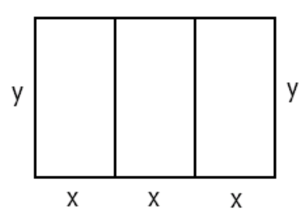


The picture shows that P will have a **relative** maximum on the interval $\left(0, \sqrt{48}\right)$ that is **also absolute maximum**. We find its x-coordinate by solving for the zero of the derivative.

$$\begin{aligned} P(b) &= -b^3 + 48b \\ P'(b) &= -3b^2 + 48 = -3\left(b^2 - 16\right) = -3(b+4)(b-4) \end{aligned}$$

Thus the maximum is at $b = 4$. Then the other number a is $48 - b^2 = 48 - 16 = 32$.

2. We have P meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure below. What is the maximal area we can enclose this way?



Solution: Let x denote the small horizontal sides, and y the vertical sides.

$$\begin{aligned} P &= 6x+4y \qquad \Longrightarrow y=\frac{1}{4}(P-6x)=-\frac{3}{2}x+\frac{P}{4} \\ A &= 3xy = 3x\left(-\frac{3}{2}x+\frac{P}{4}\right)=-\frac{9}{2}x^2+\frac{3P}{4}x \end{aligned}$$

We compute the domain of this function. Both x and y must be positive. This gives us the following inequalities: $x>0$ and $y=-\frac{3}{2}x+\frac{P}{4}>0$

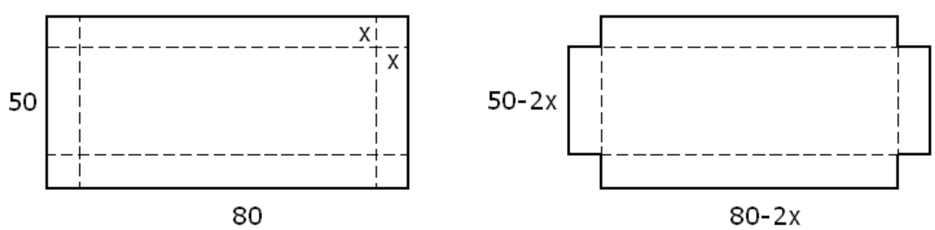
$$\begin{aligned} -\frac{3}{2}x+\frac{P}{4} &> 0 \qquad \text{solve for } x \\ \frac{P}{4} &> \frac{3}{2}x \\ \frac{P}{6} &> x \qquad \Longrightarrow \qquad \text{domain: } \left(0,\frac{P}{6}\right) \end{aligned}$$

The area function $A(x)=-\frac{9}{2}x^2+\frac{3P}{4}x$ is a downward turning parabola, so its vertex is its maximum. We can find it by completing the square or by differentiation:

$$A(x)=-\frac{9}{2}x^2+\frac{3P}{4}x=-\frac{9}{2}\left(x^2-\frac{P}{6}x\right)=-\frac{9}{2}\left(\left(x-\frac{P}{12}\right)^2-\frac{P^2}{144}\right)=-\frac{9}{2}\left(x-\frac{P}{12}\right)^2+\frac{P^2}{32}$$

When $x=\frac{P}{12}$, then the maximal area is $\frac{P^2}{32}$.

3. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?



Solution: The base of the box is a rectangle with sides $50-2x$ and $80-2x$. The height of the box is x. Thus the volume of the box, as a function of x is

$$V(x)=(80-2x)\,x\,(50-2x)=4\left(x^3-65x^2+1000x\right)$$

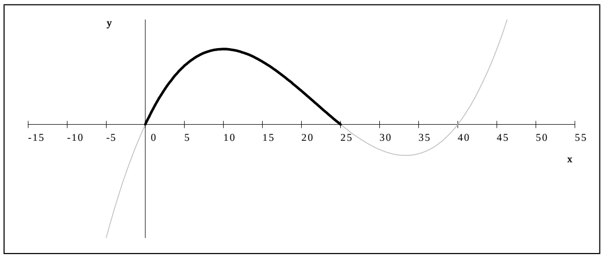
The domain is not the entire number line, clearly all sides must be positive. after we solve the inequalities $80-2x>0$ and $50-2x>0$ and $x>0$, we obtain the domain $(0,25)$. Since V is a cubic function with a positive leading coefficient, the relative extrema at the smaller $x-$value is a maximum, and the extrema at the greater $x-$value is a relative minimum. The domain is $(0,25)$.

$$V(x)=4\left(x^3-65x^2+1000x\right) \qquad \Longrightarrow \qquad V'(x)=4\left(3x^2-130x+1000\right)$$

We solve for the zeroes of the derivative

$$4\left(3x^2-130x+1000\right)=0 \qquad \Longrightarrow \quad x_1=10 \qquad x_2=\frac{100}{3}$$

The relative maximum is at $x=10$. This is an absolute maximum since the function is restricted to a domain of $(0,25)$.



So we should cut out squares with 10 centimeters long sides.

4. Prove that for any real numbers a and b, if $a+b=1$, then $a^4+b^4 \geq \frac{1}{8}$.

Solution: We will prove that the function $f(x)=x^4+(1-x)^4$ has an absolute minimum at $x=\frac{1}{2}$.

$$\begin{aligned} f(x) &= x^4+(1-x)^4 = x^4+(x-1)^4 \\ f'(x) &= 4x^3+4(x-1)^3 = 4\left(x^3+(x-1)^3\right) \end{aligned}$$

We factor via the sum of cubes theorem

$$\begin{aligned} f'(x) &= 4(x+x-1)\left(x^2-x(x-1)+(x-1)^2\right) = 4(2x-1)\left(x^2-x+1\right) \\ &= 8\left(x-\frac{1}{2}\right)\left(x^2-x+1\right) \end{aligned}$$

In case of the quadratic factor, we complete the square to see when that expression is negative.

$$8\left(x-\frac{1}{2}\right)\left(x^2-x+1\right) = 8\left(x-\frac{1}{2}\right)\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)$$

We can now see that the quadratic expression is always positive. (This is always the case with the quadratic factor in the sum or difference of two cubes.) Thus f' will be negative on $\left(-\infty,\frac{1}{2}\right)$ and positive on $\left(\frac{1}{2},\infty\right)$. Thus f is strictly decreasing on $\left(-\infty,\frac{1}{2}\right)$ and strictly increasing on $\left(\frac{1}{2},\infty\right)$. This means that f has an **absolute** minimum at $x=\frac{1}{2}$. We compute $f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^4+\left(1-\frac{1}{2}\right)^4=\frac{1}{8}$. (Note: this problem can be solved by elementary methods as well. One method involves simplifying $\left(\frac{1}{2}+x\right)^4+\left(\frac{1}{2}-x\right)^4$ and completing the square.)

5. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

Solution: Let h denote the height of the can, and r denote the radius of the base circle.

$$V=\pi r^2 h=1000 \quad \Longrightarrow \quad h=\frac{1000}{\pi r^2}$$

The domain is $(0,\infty)$

$$\begin{aligned} S(r) &= 2\pi rh+2\pi r^2 = 2\pi r\left(\frac{1000}{\pi r^2}\right)+2\pi r^2 = 2\pi r^2+\frac{2000}{r} \\ S'(r) &= 4\pi r-\frac{2000}{r^2} = \frac{4\pi r^3-2000}{r^2} \end{aligned}$$

$$\begin{aligned} 4\pi r - \frac{2000}{r^2} &= 0 \\ 4\pi r &= \frac{2000}{r^2} \\ \pi r^3 &= 500 \quad \Longrightarrow \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419\,26 \end{aligned}$$

and

$$h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi \left(500^{2/3}\right)\left(\pi^{-2/3}\right)} = \frac{2 \cdot 500}{\left(500^{2/3}\right)\left(\pi^{1/3}\right)} = \frac{2 \cdot 500^{1/3}}{\left(\pi^{1/3}\right)} = 2\sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.838\,52$$

But is this an absolute minimum we found?

$$S''(r) = 4\pi + \frac{4000}{r^3}$$

Since S'' is positive on the entire domain (recall $r > 0$), S' is strictly increasing on its entire domain. This means that S' is negative before its only zero and positive after. This implies that S is decreasing before and increasing after, and so we indeed found the absolute minimum.

