Sample Problems

1. (Section 3.5 #47) One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.

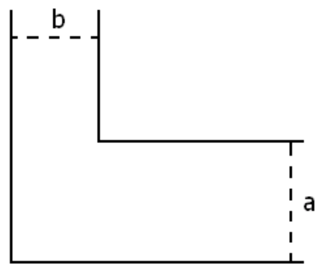
2. (Section 3.5 #49) One thousand feet of fencing is to be used to surround two areas, one square and one circular. What should the size of each area be in order that the total area be

   a) as large as possible b) as small as possible?

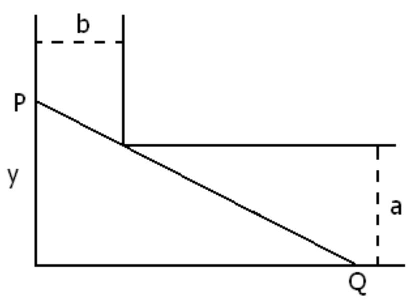
3. (Section 3.5 #39) A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?

4. (Section 3.5 #48) Find the point(s) ont the arc of the parabola $y = x^2$ for $0 \leq x \leq 1$ which are nearest to the point $(0, q)$. [Hint: minimize the square of the distance between points.]

5. (Section 3.5 #51) Consider the problem of finding the length of the longest rod we can carry through a corner shown on the picture below.



   a) For what values of y does the line segment PQ have the shortest length? (Hint: minimize the square of the length.)

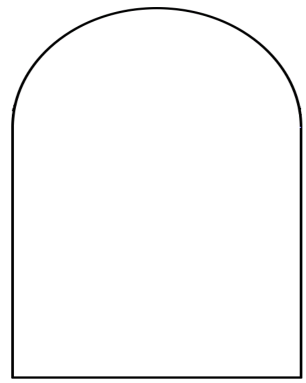


   b) What is the length of the longest ladder which can be slid along the floor from a corridor of width a to a corridor of width b?

6. (Chapter 3 Reivew, #92) Find all values of c for which the function $f(x) = \dfrac{1}{x^2+1} + cx$ is increasing on its entire domain.

7. Prove that for any real numbers a and b, if $a + b = 1$, then $a^4 + b^4 \geq \dfrac{1}{8}$.

8. A Norman window has the outline of a semicircle on top of a rectangle, as shown on the picture below. Find the dimensions of the window that can be built using 8 meters of wood and has the maximal area.



9. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?

Sample Problems - Answers

1. 4, 32

2. a) circle of radius $\frac{500}{\pi}$ feet b) the circle has radius $\frac{500}{\pi+4}$ feet, the square has sides $\frac{1000}{\pi+4}$ feet long

3. 10 cm by 10 cm

4. $(0,0)$ if $q<\frac{1}{2}$; $\left(\sqrt{q-\frac{1}{2}}, q-\frac{1}{2}\right)$ if $\frac{1}{2}\le q\le\frac{3}{2}$; and $(1,1)$ $q>\frac{3}{2}$

5. a) $y=\sqrt[3]{ab^2}+a$ b) $\left(a^{2/3}+b^{2/3}\right)^{3/2}$

6. $c\ge\frac{3\sqrt{3}}{8}$

7. see solutions

8. The dimensions are: $2x=\frac{16}{\pi+4}$ wide and $x+y=\frac{16}{\pi+4}$ tall.

9. $r=\sqrt[3]{\frac{500}{\pi}}\approx 5.419\,26$ and $h=2\sqrt[3]{\frac{500}{\pi}}=2r\approx 10.838\,521\,402\,785\,8$

Sample Problems - Solutions

1. (Section 3.5 #47) One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.
Solution: Let us denote the numbers by a and b. We have $a+b^2=48 \implies a=48-b^2$. Then the product of the two numbers is a function of b,

$$P(b)=b\left(48-b^2\right)=-b^3+48b$$

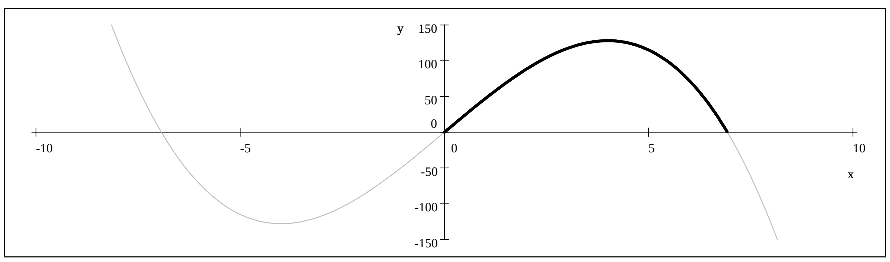
Since this is a cubic function, it does not have an absolute maximum, only if its domain is restricted. The domain is

$$b \text{ is positive } \implies b>0 \quad \text{and}$$
$$a \text{ is positive } \implies 48-b^2>0 \text{ we solve this for } b \text{ and get}$$

$$\begin{aligned} 48-b^2 &> 0 \\ 48 &> b^2 \quad \implies \quad -\sqrt{48}<b<\sqrt{48} \end{aligned}$$

which means that the domain of P is $\left(0,\sqrt{48}\right)$. (This is the set of all numbers for which both a and b are positive.) We can sketch P; it is a cubic polynomial with a negative leading coefficient, and its zeroes are at $x=0,-\sqrt{48},\sqrt{48}$ as its factored form is

$$P(b)=-b\left(b^2-48\right)=-b\left(b+\sqrt{48}\right)\left(b-\sqrt{48}\right)$$



The picture shows that P will have a **relative** maximum on the interval $\left(0,\sqrt{48}\right)$ that is **also absolute maximum**. We find its $x-$coordinate by solving for the zero of the derivative.

$$\begin{aligned} P(b) &= -b^3+48b \\ P'(b) &= -3b^2+48=-3\left(b^2-16\right)=-3(b+4)(b-4) \end{aligned}$$

Thus the maximum is at $b=4$. Then the other number a is $48-b^2=48-16=32$.

2. (Section 3.5 #49) One thousand feet of fencing is to be used to surround two areas, one square and one circular. What should the size of each area be in order that the total area be

a) as large as possible b) as small as possible?
Solution: Let us denote the radius of the circle by r. Clearly $r\geq 0$. If we denote the side of the square by s, we have an equation for the total perimeter. We can solve that for s.

$$\begin{aligned} 1000 &= 4s+2\pi r \\ \frac{100-2\pi r}{4} &= s \quad \implies \quad s=\frac{500-\pi r}{2} \end{aligned}$$

We also need $s\geq 0$

$$\begin{aligned} s &\geq 0 & 500 &\geq \pi r \\ \frac{500-\pi r}{2} &\geq 0 & \frac{500}{\pi} &\geq r \\ 500-\pi r &\geq 0 \end{aligned}$$

Thus the area-function, (yet to be set up) will have domain $\left[0, \dfrac{500}{\pi}\right]$. The endpoints mean that all 1000 feet was used for the square (in case of $r = 0$) or for the circle (in case of $r = \dfrac{500}{\pi}$). The area, as a function of r is now

$$A(r) = \pi r^2 + s^2 = \pi r^2 + \left(\frac{500 - \pi r}{2}\right)^2 = \left(\pi + \frac{1}{4}\pi^2\right) r^2 - 250\pi r + 62\,500$$

Since this is a quadratic expression with a positive leading coefficient, we know that the vertex is a minimum, both relative and absolute. The vertex can be found by either elementary methods or by finding the zero of $A'(r)$. Either way, we will get that $r_{\min} = \dfrac{500}{4+\pi}$. This means that the circle will have radius $r = \dfrac{500}{4+\pi}$ and the square will have sides $s_{\min} = \dfrac{500 - \pi r_{\min}}{2} = \dfrac{1000}{\pi + 4}$ since

$$\begin{aligned} s_{\min} &= \frac{500 - \pi r_{\min}}{2} = \frac{500 - \pi\left(\frac{500}{4+\pi}\right)}{2} = s_{\min} = \frac{500 - \pi\left(\frac{500}{4+\pi}\right)}{2} \cdot \frac{4+\pi}{4+\pi} \\ &= \frac{500(4+\pi) - 500\pi}{2(4+\pi)} = \frac{2000 + 500\pi - 500\pi}{2(4+\pi)} = \frac{2000}{2(4+\pi)} = \frac{1000}{\pi+4} \end{aligned}$$

These dimensions give us the minimal area

$$\begin{aligned} A_{\min} &= \pi r_{\min}^2 + s_{\min}^2 \\ &= \pi\left(\frac{500}{4+\pi}\right)^2 + \left(\frac{1000}{\pi+4}\right)^2 = \frac{250\,000}{\pi+4} \end{aligned}$$

But what about the maximum? A quadratic expression does not have a relative maximum if its leading coefficient is positive. The absolute maximum must be at one of the end-points of the interval $\left[0, \dfrac{500}{\pi}\right]$. We need to compare these areas. When $r = 0$, all thousand feet was used for the square. The area is then

$$A(0) = 250^2$$

or, using our expression for the area,

$$A(r) = \left(\pi + \frac{1}{4}\pi^2\right) r^2 - 250\pi r + 62\,500 \quad \Longrightarrow A(0) = 62\,500$$

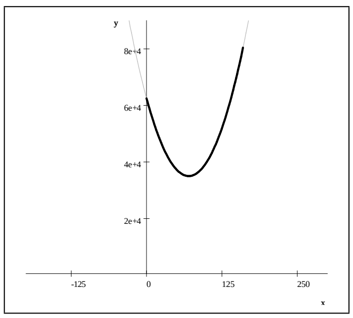
When $r = \dfrac{500}{\pi}$, all thousand feet was used for the circle. Then the area is

$$A\left(\frac{500}{\pi}\right) = \pi r^2 = \pi\left(\frac{500}{\pi}\right)^2 = \frac{500^2}{\pi}$$

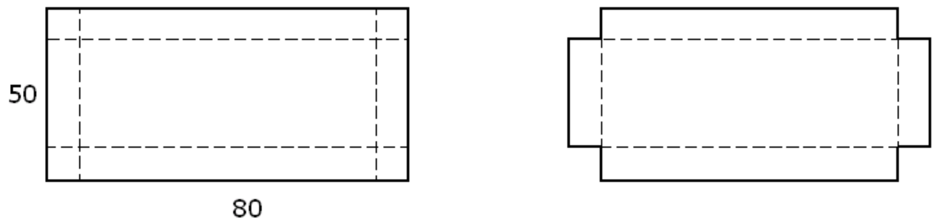
Since $\dfrac{500^2}{\pi} = 79\,577.471\,545\,5$,

$$A(0) < A\left(\frac{500}{\pi}\right)$$

which means that the greatest area occurs if we use all thousand feet for the circle.



3. (Section 3.5 #39) A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?



Solution: The base of the box is a rectangle with sides $50 - 2x$ and $80 - 2x$. The height of the box is x. Thus the volume of the box, as a function of x is

$$V(x) = (80 - 2x)\, x\, (50 - 2x) = 4\left(x^3 - 65x^2 + 1000x\right)$$

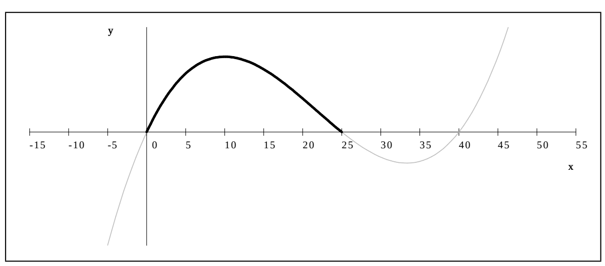
The domain is not the entire number line, clearly all sides must be positive. after we solve the inequalities $80 - 2x > 0$ and $50 - 2x > 0$ and $x > 0$, we obtain the domain $(0, 25)$. Since V is a cubic function with a positive leading coefficient, the relative extrema at the smaller $x-$value is a maximum, and the extrema at the greater $x-$value is a relative minimum. The domain is $(0, 25)$.

$$V(x) = 4\left(x^3 - 65x^2 + 1000x\right) \quad \Longrightarrow \quad V'(x) = 4\left(3x^2 - 130x + 1000\right)$$

We solve for the zeroes of the derivative

$$4\left(3x^2 - 130x + 1000\right) = 0 \quad \Longrightarrow \quad x_1 = 10 \qquad x_2 = \frac{100}{3}$$

The relative maximum is at $x = 10$. This is an absolute maximum since the function is restricted to a domain of $(0, 25)$.



So we should cut out squares with 10 centimeters long sides.

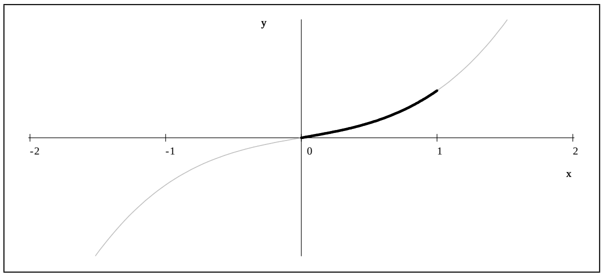
4. (Section 3.5 #48) Find the point(s) ont the arc of the parabola $y = x^2$ for $0 \le x \le 1$ which are nearest to the point $(0, q)$. [Hint: minimize the square of the distance between points.]
Solution: Let $f(x)$ denote the square of the distance between the point $\left(x, x^2\right)$ and $(0, q)$.

$$f(x) = (x - 0)^2 + \left(x^2 - q\right)^2 = x^2 + \left(q - x^2\right)^2 = x^4 + x^2(1 - 2q) + q^2$$

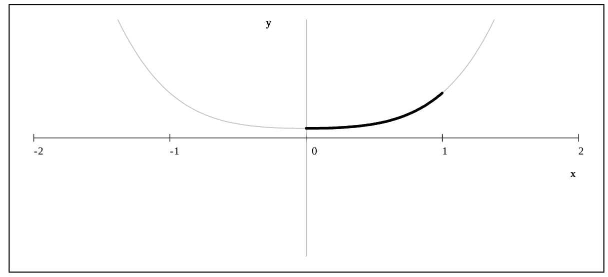
We differentiate:

$$f'(x) = 4x^3 + 2x(1 - 2q) = 4x\left(x^2 - \left(q - \frac{1}{2}\right)\right)$$

Case 1. If $q - \frac{1}{2} < 0$, the expression $4x\left(x^2 - \left(q - \frac{1}{2}\right)\right)$ is positive on $[0, 1]$ and so f is a strictly increasing function. Then $(0, f(0))$ is an absolute minimum. This means that if $q < \frac{1}{2}$, then $(0, 0)$ is the point closest to $(0, q)$



Graph of f'

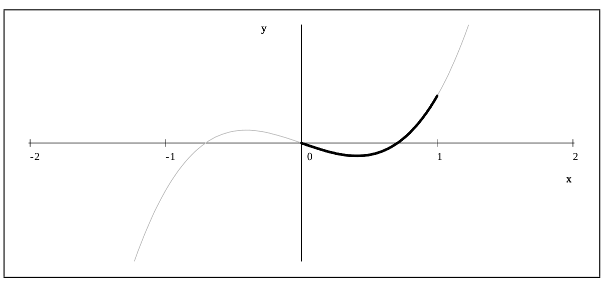


Graph of f

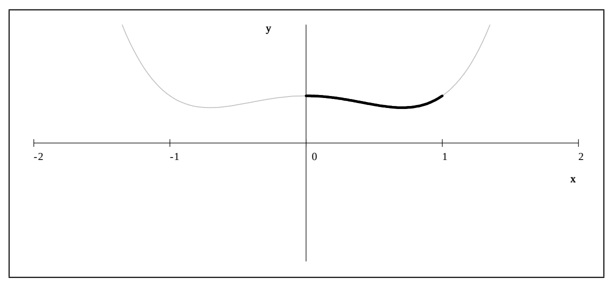
Case 2. If $\frac{1}{2} \le q \le \frac{3}{2}$, then $f'(x)$ factors via the difference of squares theorem:

$$f'(x) = 4\left(x + \sqrt{q - \frac{1}{2}}\right) x \left(x - \sqrt{q - \frac{1}{2}}\right)$$

Since f' is a cubic function with a positive lead ing coefficient, f has a relative minimum at $x = -\sqrt{q - \frac{1}{2}}$ and $x = \sqrt{q - \frac{1}{2}}$; and a relative maximum at $x = 0$. Tha absolute maximum on $[0, 1]$ is when $x = \sqrt{q - \frac{1}{2}}$. Then the point is $\left(\sqrt{q - \frac{1}{2}}, q - \frac{1}{2}\right)$.



Graph of f'

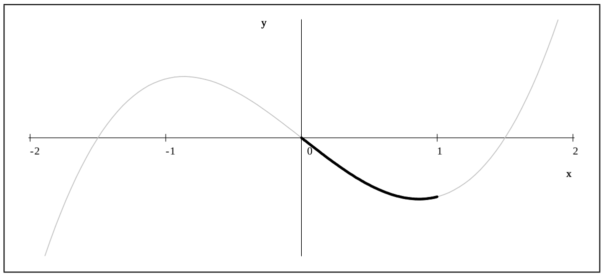


Graph of f

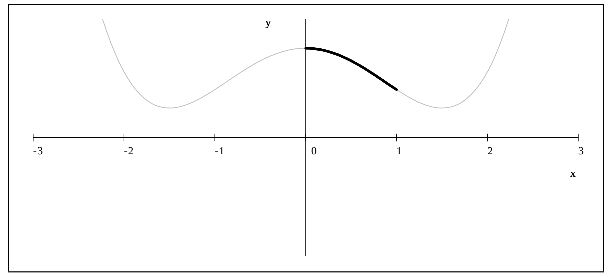
Case 3. If $q > \frac{3}{2}$. then $f'(x)$ factors via the difference of squares theorem as before

$$f'(x) = 4\left(x + \sqrt{q - \frac{1}{2}}\right) x \left(x - \sqrt{q - \frac{1}{2}}\right)$$

and f has a relative minimum at $x = -\sqrt{q - \frac{1}{2}}$ and $x = \sqrt{q - \frac{1}{2}}$; and a relative maximum at $x = 0$. However, none of these values lie within the interval $[0, 1]$. Consequently, f' is negative on the entire domain $[0, 1]$ and so f is strictly decreasing on $[0, 1]$. Then the lowest value is at the other endpoint, at $x = 1$. Then $(1, 1)$ is the closest point.

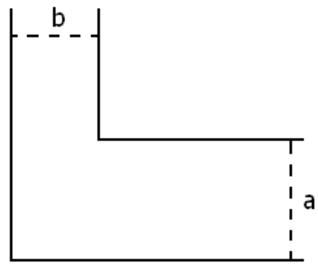


Graph of f'

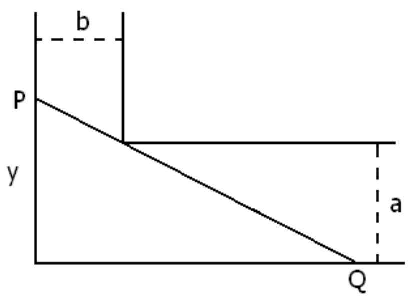


Graph of f

5. (Section 3.5 #51) Consider the problem of finding the length of the longest rod we can carry through a corner shown on the picture below.



a) For what values of y does the line segment PQ have the shortest length? (Hint: minimize the square of the length.)



Solution: Let x denote the horizontal side of the right triangle with hypotenuse PQ. Using similar trianlges,

$$\frac{y-a}{b} = \frac{y}{x} \qquad \Longrightarrow x = \frac{by}{y-a}$$

The square of the length is (by the Pythagorean Theorem) is on domain $(0, \infty)$

$$\begin{aligned} f(y) &= x^2 + y^2 = \left(\frac{by}{y-a}\right)^2 + y^2 = y^2 + b^2\left(\frac{y}{y-a}\right)^2 \\ f'(y) &= \left(y^2 + b^2\frac{y^2}{(y-a)^2}\right)' = 2y - \frac{2ab^2 y}{(y-a)^3} \end{aligned}$$

$$\begin{aligned} f'(y) &= 0 \\ 2y - \frac{2ab^2 y}{(y-a)^3} &= 0 \\ 2y\left(1 - \frac{ab^2}{(y-a)^3}\right) &= 0 \quad \Longrightarrow \quad y = 0 \quad \text{or} \quad 1 - \frac{ab^2}{(y-a)^3} = 0 \end{aligned}$$

$$1-\frac{ab^2}{(y-a)^3} = 0 \qquad \sqrt[3]{ab^2} = y-a$$

$$\frac{ab^2}{(y-a)^3} = 1 \qquad y = \sqrt[3]{ab^2}+a$$

$$ab^2 = (y-a)^3$$

We need to make sure that $y=\sqrt[3]{ab^2}+a$ is indeed an absolute minimum.

b) What is the length of the longest ladder which can be slid along the floor from a corridor of width a to a corridor of width b?

Solution: We simply need to evaluate $L(y)=\sqrt{f(y)}$ at $y=\sqrt[3]{ab^2}+a=a^{1/3}b^{2/3}+a$

$$\sqrt{f(y)} = \sqrt{y^2+b^2\left(\frac{y}{y-a}\right)^2} = y\sqrt{1+\frac{b^2}{(y-a)^2}}$$

$$\begin{aligned}
\sqrt{f\left(a^{1/3}b^{2/3}+a\right)} &= \left(a^{1/3}b^{2/3}+a\right)\sqrt{1+\frac{b^2}{\left(a^{1/3}b^{2/3}+a-a\right)^2}} = \left(a^{1/3}b^{2/3}+a\right)\sqrt{1+\frac{b^2}{\left(a^{1/3}b^{2/3}\right)^2}} \\
&= a^{1/3}\left(b^{2/3}+a^{2/3}\right)\sqrt{1+\frac{b^2}{a^{2/3}b^{4/3}}} = \left(b^{2/3}+a^{2/3}\right)\sqrt{a^{2/3}\left(1+\frac{b^{2-4/3}}{a^{2/3}}\right)} \\
&= \left(b^{2/3}+a^{2/3}\right)\sqrt{a^{2/3}\left(1+\frac{b^{2/3}}{a^{2/3}}\right)} = \left(b^{2/3}+a^{2/3}\right)\sqrt{a^{2/3}+b^{2/3}} \\
&= \left(a^{2/3}+b^{2/3}\right)^{3/2}
\end{aligned}$$

6. Find all values of c for which the function $f(x)=\dfrac{1}{x^2+1}+cx$ is increasing on its entire domain.

Solution: $f(x)$ is always increasing if $f'(x)$ is always positive. $f'(x)=\dfrac{-2x}{(x^2+1)^2}+c$. We will use calculus to prove that the absolute minimum of $\dfrac{-2x}{(x^2+1)^2}$ is $-\dfrac{3\sqrt{3}}{8}$ and so if we set $c=-\dfrac{3\sqrt{3}}{8}$, then $f'(x)=\dfrac{-2x}{(x^2+1)^2}+c$ will be non-negative for all x.

Define $g(x)=\dfrac{-2x}{(x^2+1)^2}$

We will prove the following statements:

1) $f'(x)=g(x)+c$

2) $g(x)$ has an absolute minimum: $\left(\dfrac{\sqrt{3}}{3},-\dfrac{3\sqrt{3}}{8}\right) \implies g(x)\geq -\dfrac{3\sqrt{3}}{8}$

$\implies$ 3) To make f' non-negative on $\mathbb{R}$, we need to add at least $\dfrac{3\sqrt{3}}{8}$. Thus $c\geq\dfrac{3\sqrt{3}}{8}$

We differentiate $f(x)=\dfrac{1}{x^2+1}+cx$ and obtain $f(x)=\dfrac{-2x}{x^2+1}+c=g(x)+c$

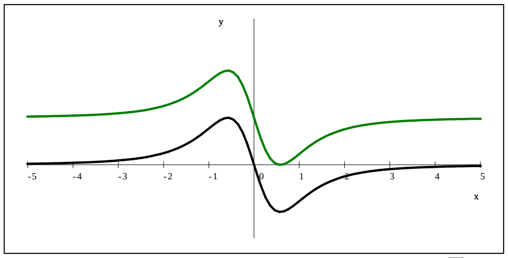
Consider $g(x)=\dfrac{-2x}{(x^2+1)^2}$. Since the denominator is always positive, we can easily see that $g(x)$ is positive on $(-\infty,0)$ and negative on $(0,\infty)$.

We compute $g'(x)=\dfrac{6x^2-2}{(x^2+1)^3}=\dfrac{6\left(x^2-\frac{1}{3}\right)}{(x^2+1)^3}=\dfrac{6\left(x+\frac{1}{\sqrt{3}}\right)\left(x-\frac{1}{\sqrt{3}}\right)}{(x^2+1)^3}$

The denominator is always positive, the numerator is a quadratic expression with a positive leading coefficient, and so g' is positive on $\left(-\infty,-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}},\infty\right)$ and negative on $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$. Consequently, g is increasing on $\left(-\infty,-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}},\infty\right)$ and decreasing on $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$; it has a relative maximum at $x=-\frac{1}{\sqrt{3}}$ and a relative minimum at $x=\frac{1}{\sqrt{3}}$.

We evaluate $g\left(\frac{1}{\sqrt{3}}\right)=\dfrac{-2\left(\frac{1}{\sqrt{3}}\right)}{\left(\left(\frac{1}{\sqrt{3}}\right)^2+1\right)^2}=-\frac{3\sqrt{3}}{8}$.

We now claim that $\left(\frac{\sqrt{3}}{3},-\frac{3\sqrt{3}}{8}\right)$ is an **absolute** minimum of $g(x)$. Since $g(x)$ is decreasing on $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ and increasing on $\left(\frac{1}{\sqrt{3}},\infty\right)$, this point is clearly below any point of g on $\left(-\frac{1}{\sqrt{3}},\infty\right)$. And if $x<-\frac{1}{\sqrt{3}}$, then $g(x)$ is positive and thus greater than $-\frac{3\sqrt{3}}{8}$.



Graphs of $g(x)$ and $g(x)+\frac{3\sqrt{3}}{8}$

7. Prove that for any real numbers a and b, if $a+b=1$, then $a^4+b^4\geq\frac{1}{8}$.

Solution: We will prove that the function $f(x)=x^4+(1-x)^4$ has an absolute minimum at $x=\frac{1}{2}$.

$$\begin{aligned} f(x) &= x^4+(1-x)^4=x^4+(x-1)^4 \\ f'(x) &= 4x^3+4(x-1)^3=4\left(x^3+(x-1)^3\right) \end{aligned}$$

We factor via the sum of cubes theorem

$$\begin{aligned} f'(x) &= 4(x+x-1)\left(x^2-x(x-1)+(x-1)^2\right)=4(2x-1)\left(x^2-x+1\right) \\ &= 8\left(x-\frac{1}{2}\right)\left(x^2-x+1\right) \end{aligned}$$

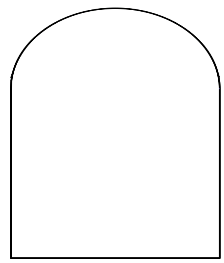
In case of the quadratic factor, we complete the square to see when that expression is negative.

$$8\left(x-\frac{1}{2}\right)\left(x^2-x+1\right)=8\left(x-\frac{1}{2}\right)\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)$$

We can now see that the quadratic expression is always positive. (This is always the case with the quadratic factor in the sum or difference of two cubes.) Thus f' will be negative on $\left(-\infty,\frac{1}{2}\right)$ and positive on $\left(\frac{1}{2},\infty\right)$. Thus f is strictly decreasing on $\left(-\infty,\frac{1}{2}\right)$ and strictly increasing on $\left(\frac{1}{2},\infty\right)$. This means that f has an

absolute minimum at $x = \frac{1}{2}$. We compute $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 + \left(1 - \frac{1}{2}\right)^4 = \frac{1}{8}$. (Note: this problem can be solved by elementary methods as well. One method involves simplifying $\left(\frac{1}{2} + x\right)^4 + \left(\frac{1}{2} - x\right)^4$ and completing the square.)

8. A Norman window has the outline of a semicircle on top of a rectangle, as shown on the picture below. Find the dimensions of the window that can be built using 8 meters of wood and has the maximal area.



Solution: Let x denote half of the bottom side. (Also the radius.)
Let y denote the vertical side. Then $2x + 2y + \pi x = 8$. We solve for y and obtain

$$y = -\left(\frac{\pi + 2}{2}\right) x + 4$$

The domain will be determined by the conditions $x > 0$ and $y \geq 0$. The domain is $\left(0, \frac{8}{\pi + 2}\right]$.We set up the area-function:

$$A(x) = 2xy + \frac{\pi x^2}{2} = 2x\left(-\left(\frac{\pi + 2}{2}\right) x + 4\right) + \frac{\pi x^2}{2} = \left(-\frac{\pi}{2} - 2\right) x^2 + 8x$$

The vertex of this upside down parabola is at $x = \frac{8}{\pi + 4}$, which is in the domain. Then $y = \frac{8}{\pi + 4}$. The dimensions are: $2x = \frac{16}{\pi + 4}$ wide and $x + y = \frac{16}{\pi + 4}$ tall.

9. A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cubic centimeters (one liter). What dimensions would guarantee the minimal amount of aluminum needed to produce a can?
Solution: Let h denote the height of the can, and r denote the radius of the base circle.

$$\pi r^2 h = 1000 \qquad h = \frac{1000}{\pi r^2}$$

The domain is $(0, \infty)$

$$\begin{aligned} S(r) &= 2\pi r h + 2\pi r^2 = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r} \\ S'(r) &= 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} \end{aligned}$$

$$\begin{aligned} 4\pi r - \frac{2000}{r^2} &= 0 \\ 4\pi r &= \frac{2000}{r^2} \\ \pi r^3 &= 500 \quad \Longrightarrow \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.41926 \end{aligned}$$

and

$$h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}} \right)^2} = \frac{1000}{\pi \left(500^{2/3} \right) \left(\pi^{-2/3} \right)} = \frac{2 \cdot 500}{\left(500^{2/3} \right) \left(\pi^{1/3} \right)} = \frac{2 \cdot 500^{1/3}}{\left(\pi^{1/3} \right)} = 2 \sqrt[3]{\frac{500}{\pi}} = 2r \approx 10.838\,52$$

But is this an absolute minimum we found?

$$S''(r) = 4\pi + \frac{4000}{r^3}$$

Since S'' is positive on the entire domain (recall $r > 0$), S' is strictly increasing on its entire domain. This means that S' is negative before its only zero and positive after. This implies that S is decreasing before and increasing after, and so we indeed found the absolute minimum.

