1. Sketch the curve defined by the parametric equation given.

a) $x = t \qquad y = t^2 \qquad -3 \leq t \leq 3$

b) $x = -t^2 \qquad y = 2t \qquad t \in \mathbb{R}$

c) $x = 3\cos t \qquad y = 3\sin t \qquad 0 \leq t \leq \pi$

d) $x = 2\cos t \qquad y = 3\sin t \qquad 0 \leq t \leq 2\pi$

e) $x = \sec t \qquad y = \tan t \qquad -\frac{\pi}{4} < t < \frac{\pi}{4}.$

f) $x = \sqrt{t+1} \qquad y = \sqrt{t} \qquad t \geq 0$

2. Find the rectangular equation of each of the curves defined by the parametric equation given.

a) $x = t \qquad y = t^2 \qquad -3 \leq t \leq 3$

b) $x = -t^2 \qquad y = 2t \qquad t \in \mathbb{R}$

c) $x = 3\cos t \qquad y = 3\sin t \qquad 0 \leq t \leq \pi$

d) $x = 2\cos t \qquad y = 3\sin t \qquad 0 \leq t \leq 2\pi$

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f) $x = \sqrt{t+1} \qquad y = \sqrt{t} \qquad t \geq 0$

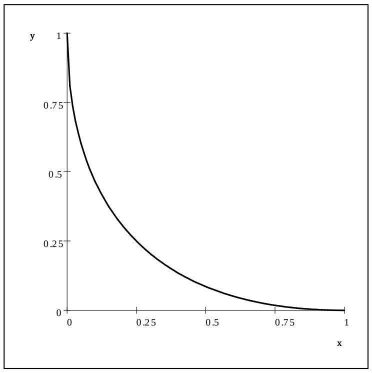
3. Find a parametrization for the line $y = 3x - 1$.

4. Find a parametization for the line with slope m that passes through the point (a, b).

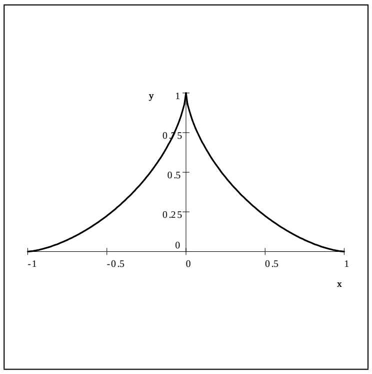
5. Compute the tangent line to the curve $x = t - t^2 \qquad y = t - t^3$ at the point $x = (0, 0)$, where $t = 1$.

6. Compute the tangent line to the curve $x = \sec t \qquad y = \tan t \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$ at the point $x = \left(\sqrt{2}, 1\right)$, where $t = \frac{\pi}{4}$.

7. Find the area under the curve defined by $x = \cos^4 t \quad y = \sin^4 t \quad t \in \left[0, \frac{\pi}{2}\right]$



8. Find the length of the curve defined by $x = \cos^3 t$ and $y = \sin^3 t \quad t \in [0, \pi]$

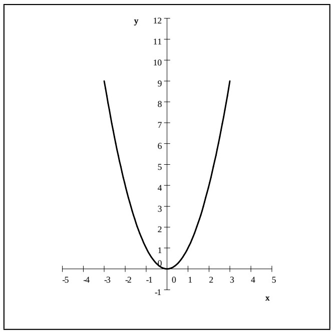


9. Find the length of the curve defined by $x = \cos t$ and $y = \sin t \quad t \in [0, \pi]$

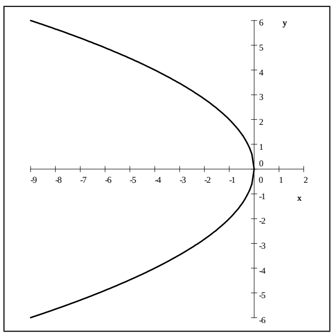
Answers

1. Sketch the curve defined by the parametric equation given.

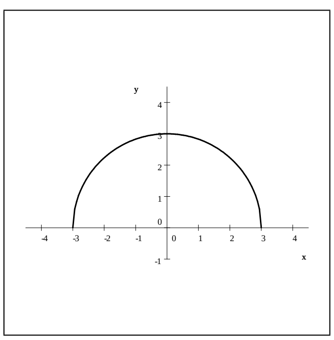
a) $x = t \qquad y = t^2 \qquad -3 \le t \le 3$



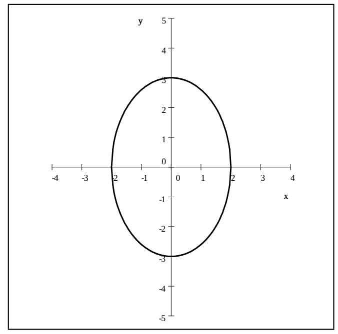
b) $x = -t^2 \qquad y = 2t \qquad t \in \mathbb{R}$



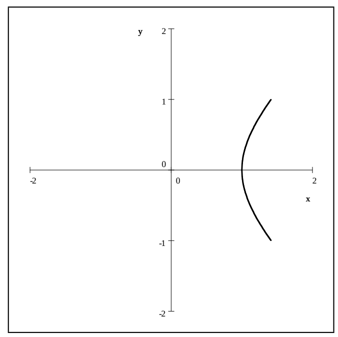
c) $x = 3\cos t \qquad y = 3\sin t \qquad 0 \le t \le \pi$



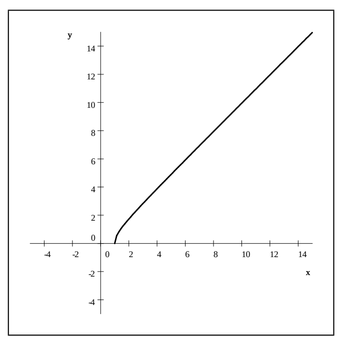
d) $x = 2\cos t \qquad y = 3\sin t \qquad 0 \le t \le 2\pi$



e) $x = \sec t \qquad y = \tan t \qquad -\dfrac{\pi}{4} < t < \dfrac{\pi}{4}.$



f) $x = \sqrt{t+1} \qquad y = \sqrt{t} \qquad t \ge 0$



2. Find the regular equation of each of the curves defined by the parametric equation given.

a) $x = t \qquad y = t^2 \qquad -3 \le t \le 3$

$y = x^2$ on $[-3, 3]$

b) $x = -t^2 \qquad y = 2t \qquad t \in \mathbb{R}$

$x = -\dfrac{y^2}{4} \qquad y \in \mathbb{R}$

c) $x = 3\cos t \qquad y = 3\sin t \qquad 0 \le t \le \pi$

$y = \sqrt{9 - x^2} \qquad x \in [-3, 3]$

d) $x = 2\cos t \qquad y = 3\sin t \qquad 0 \le t \le 2\pi$

$\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \qquad x \in [-2, 2]$

e) $x = \sec t \qquad y = \tan t \qquad -\dfrac{\pi}{4} < t < \dfrac{\pi}{4}.$

$x^2 - y^2 = 1 \qquad 0 < x < \sqrt{2} \qquad -1 < y < 1$

f) $x = \sqrt{t+1} \qquad y = \sqrt{t} \qquad t \ge 0$

$x^2 - y^2 = 1 \qquad x \ge 1, \quad y \ge 0$

3. Find a parametrization for the line $y = 3x - 1$.

   answers may vary $x = t \quad y = 3t - 1$

4. Find a parametization for the line with slope m that passes through the point (a, b).

   answers may vary $x = t \qquad y = m(t - a) + b$

5. Compute the tangent line to the curve $x = t - t^2 \qquad y = t - t^3$ at the point $x = (0, 0)$, where $t = 1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(t - t^3\right)}{\frac{d}{dt}\left(t - t^2\right)} = \frac{1 - 3t^2}{1 - 2t}$$

   At $t = 1$, the slope is $\frac{1-3}{1-2} = 2$. So the tangent line is $y = 2x$.

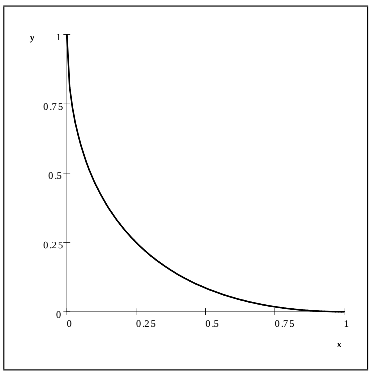
6. Compute the tangent line to the curve $x = \sec t \qquad y = \tan t \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$ at the point $x = \left(\sqrt{2}, 1\right)$, where $t = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\tan t)}{\frac{d}{dt}(\sec t)} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \frac{1}{\cos t} \cdot \frac{\cos t}{\sin t} = \frac{1}{\sin t}$$

   at $t = \frac{\pi}{4}$, that is $\frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$. Now that we know the slope is $m = \sqrt{2}$ and the tangent line passes through $\left(\sqrt{2}, 1\right)$, the equation is

$$\begin{aligned} y - 1 &= \sqrt{2}\left(x - \sqrt{2}\right) \\ y - 1 &= \sqrt{2}x - 2 \\ y &= \sqrt{2}x - 1 \end{aligned}$$

7. Find the area under the curve defined by $x = \cos^4 t \quad y = \sin^4 t \qquad t \in \left[0, \frac{\pi}{2}\right]$



   At first, we could say that the area is the integral

$$A = \int_0^1 y\,dx$$

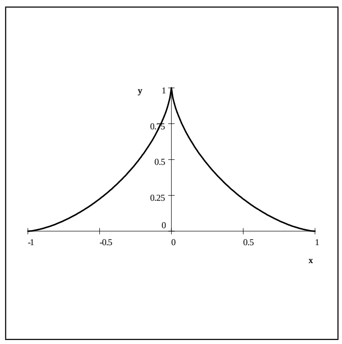
and then we express everything in terms of t. From $x = \cos^4 t$ we obtain that $dx = 4\left(\cos^3 t\right)(-\sin t)\,dt = -4\cos^3 t \sin t dt$ and so the integral becomes

$$\begin{aligned} A &= \int_0^1 y dx = \int_{\pi/2}^0 \sin^4 t \ \left(-4\cos^3 t \sin t \ dt\right) = \int_{\pi/2}^0 -4\cos^3 t \sin^5 t \ dt = \int_{\pi/2}^0 -4\cos t\left(\cos^2 t\right)\sin^5 t \ dt \\ &= \int_{\pi/2}^0 -4\cos t\left(1-\sin^2 t\right)\sin^5 t \ dt = \int_{\pi/2}^0 -4\cos t\left(\sin^5 t - \sin^7 t\right) \ dt \end{aligned}$$

Let $u = \sin t$ then $du = \cos t dt$. When $t = \dfrac{\pi}{2}$, then $u = 1$ and when $t = 0$, then $u = 0$.

$$\begin{aligned} A &= \int_{\pi/2}^0 -4\cos t\left(\sin^5 t - \sin^7 t\right) \ dt = \int_1^0 -4\left(u^5 - u^7\right) \ dt = -4\left(\frac{u^6}{6} - \frac{u^8}{8}\Big|_1^0\right) \\ &= -4\left(0 - \left(\frac{1}{6} - \frac{1}{8}\right)\right) = -4\left(-\frac{1}{24}\right) = \frac{1}{6} \end{aligned}$$

8. Find the length of the curve defined by $x = \cos^3 t$ and $y = \sin^3 t$ $\ t \in [0, \pi]$



By symmetry, we will compute the arc length in the first quadrant only, and then multiply the result by 2.

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$\dfrac{dx}{dt} = \dfrac{d}{dt}\left(\cos^3 t\right) = 3\cos^2 t\,(-\sin t) = -3\cos^2 t \sin t$ and $\dfrac{dy}{dt} = \dfrac{d}{dt}\left(\sin^3 t\right) = 3\sin^2 t\,(\cos t) = 3\sin^2 t \cos t$

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{\pi/2} \sqrt{\left(-3\cos^2 t \sin t\right)^2 + \left(3\sin^2 t \cos t\right)^2} dt \\ &= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt = \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t\left(\sin^2 t + \cos^2 t\right)} dt \\ &= \int_0^{\pi/2} \left|3\cos t \sin t\right| \sqrt{\cos^2 t + \sin^2 t} dt = \int_0^{\pi/2} 3\cos t \sin t dt = \int_0^{\pi/2} \frac{3}{2}\sin 2t dt = -\frac{3}{4}\cos 2t\Big|_0^{\pi/2} \\ &= -\frac{3}{4}\left(\cos\left(2\cdot\frac{\pi}{2}\right) - \cos(2\cdot 0)\right) = -\frac{3}{4}(\cos \pi - \cos 0) = -\frac{3}{4}(-2) = \frac{3}{2} \end{aligned}$$

So the length of the full arc on $[0, \pi]$ is 3.

9. Find the length of the curve defined by $x = \cos t$ and $y = \sin t \quad t \in [0, \pi]$

$\dfrac{dx}{dt} = \dfrac{d}{dt}(\cos t) = -\sin t$ and $\dfrac{dy}{dt} = \dfrac{d}{dt}(\sin t) = \cos t$

$$L = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^\pi \sqrt{\cos^2 t + \sin^2 t} dt = \int_0^\pi \sqrt{1} dt = \int_0^\pi 1 dt = t\Big|_0^\pi = \pi$$

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