The real numbers has the **completeness property**: If a non-empty set of real numbers is bounded above, then there exists a least upper bound; if a non-empty set of real numbers is bounded below, then there exists a greatest lower bound. (This is an axiom of the real numbers, and this is the one that distinguishes the set of rational numbers from the set of real numbers.)

Definition: A **sequence** is a list of numbers $a_1, a_2, a_3, ..., a_n, ...$ in a given order. The numbers a_n are **terms** of the sequence. The integer k is called the index of the term a_k.

An infinite sequence is a function with domain $\mathbb{N}$. We may start labeling at a number greater than 1.

We can describe sequences by writing rules

$$a_n = \sqrt{n} \quad b_n = (-1)^{n+1}\frac{1}{n} \quad c_n = \frac{n-1}{n} \quad d_n = (-1)^{n+1}$$

or listing the first few terms

$$\{a_n\} = \left\{1, \sqrt{2}, \sqrt{3}, ..., \sqrt{n}, ...\right\} \qquad \{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ..., \frac{n-1}{n}, ...\right\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, ..., (-1)^{n+1}\frac{1}{n}, ...\right\} \qquad \{d_n\} = \{1, -1, 1-1, ..., (-1), ...\}$$

Definition: The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there exists an integer N such that for all n,

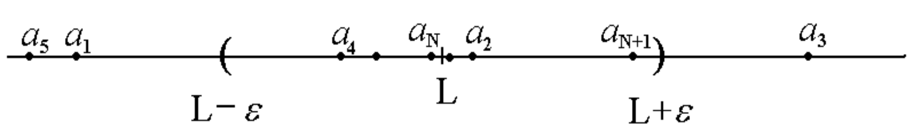
$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon.$$

If no such number L exists, we say $\{a_n\}$ **diverges**.
If $\{a_n\}$ converges to L, we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ and call L the **limit** of the sequence.

Notice that $|a_n - L| < \varepsilon$ and $L - \varepsilon < a_n < L + \varepsilon$ are equivalent statements.

$$\begin{aligned} L - \varepsilon &< a_n < L + \varepsilon \\ -\varepsilon &< a_n - L < \varepsilon \\ |a_n - L| &< \varepsilon \end{aligned}$$

This definition is fundamental to our material. We can think of a convergent sequences as one whose elements are eventually arbitrarily close to its limit. For any positive value of ε (think of ε as the error), all but finitely many elements (think of them as the first few) of the sequence are inside an $\varepsilon-$neighborhood of L.



If the sequence $\{a_n\}$ converges to L, then no matter how small ε is, all but a finitely many terms of the sequence will fall inside the ε-neightborhood of L. For every value of ε, there is generally a different value of N.

Definition: the symbols $\lceil$ and $\rceil$ denote the **ceiling function**: for any real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. For example, $\lceil 5 \rceil = 5$, $\lceil 17.34 \rceil = 18$, and $\lceil -2.37 \rceil = -2$.

We similarly define the **floor function**: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, $\lfloor 5 \rfloor = 5$, $\lfloor 17.34 \rfloor = 17$, and $\lfloor -2.37 \rfloor = -3$.

The floor and ceiling functions are useful when we need to start with a real number such as $\frac{1}{\varepsilon}$ and turn it into an integer such as N.

Before we see some examples, let us establish a fact that we will use very often.

Theorem: If A and B are real numbers such that they are either both positive or both negative, then

$$A < B \text{ implies } \frac{1}{A} > \frac{1}{B}$$

This theorem allows us to (carefully) take the reciprocal of both sides in an inequality. If both sides are of the same sign, we can take the reciprocal of both sides but must reverse the inequality sign. (We will prove this theorem in the exercises.)

Example 1. The constant sequence $\{a_n\} = \{c, c, c, ...\}$. Clearly $\lim_{n\to\infty} a_n = c$.
proof: Let $\varepsilon > 0$ be given. Then $N = 1$ will do, because for all $n > 1$ we will have that $|c - c| = 0 < \varepsilon$. That is the same as $|a_n - c| < \varepsilon$ and so the sequence converges to c.

Example 2. $\lim_{n\to\infty} \frac{1}{n} = 0$
proof: Let $\varepsilon > 0$ be given. Define $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$.

If $n > N$, then $n > N > \frac{1}{\varepsilon}$ and so $\frac{1}{n} < \varepsilon$. Since $\frac{1}{n}$ is positive, we have that

$$\begin{aligned} -\varepsilon &< \frac{1}{n} < \varepsilon \\ -\varepsilon &< \frac{1}{n} - 0 < \varepsilon \quad \text{same as} \quad \left|\frac{1}{n} - 0\right| < \varepsilon \end{aligned}$$

and so $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 3. The sequence defined $a_n = \frac{3n^3 + 2}{n^3}$ approach 3. We will give an $\varepsilon - N$ proof.

proof: Since $\frac{3n^3 + 2}{n^3} = 3 + \frac{2}{n^3}$, as n becomes larger and larger, $\frac{2}{n^3}$ will approach zero and so a_n will approach 3. We will prove that 3 is the limit.

Let $\varepsilon > 0$ be given. Let N be a positive integer with $N \geq \frac{2}{\varepsilon}$. For all $n > N$, we have that

$$\begin{aligned} n &> \frac{2}{\varepsilon} \\ n^3 &> n > \frac{2}{\varepsilon} \\ n^3 &> \frac{2}{\varepsilon} \qquad \text{both sides are positive} \\ \frac{1}{n^3} &< \frac{\varepsilon}{2} \\ \frac{2}{n^3} &< \varepsilon \end{aligned}$$

also, $\frac{2}{n^3} > 0$ and since $-\varepsilon < 0$, we also have that

$$\begin{aligned} -\varepsilon &< \frac{2}{n^3} < \varepsilon \\ -\varepsilon &< \frac{2}{n^3} + 3 - 3 < \varepsilon \\ -\varepsilon &< \frac{3n^3+2}{n^3} - 3 < \varepsilon \quad \text{same as} \quad \left| \frac{3n^3+2}{n^3} - 3 \right| < \varepsilon \\ -\varepsilon &< a_n - 3 < \varepsilon \quad \text{same as} \quad |a_n - 3| < \varepsilon \end{aligned}$$

and so the sequence converges to 3.

Example 4. $\{1, \ -1, \ 1, \ -1, \ 1, \ -1, \ ....\}$ diverges.

proof: Suppose for a contradiction that such a number L exists. Let $\varepsilon = \frac{1}{3}$.

There exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \frac{1}{3}$. Since 1 occurs in the sequence at arbitrarily high index, it must be that

$$\begin{aligned} |1 - L| &< \frac{1}{3} \\ -\frac{1}{3} &< L - 1 < \frac{1}{3} \\ \frac{2}{3} &< L < \frac{4}{3} \end{aligned}$$

Since -1 occurs in the sequence at arbitrarily high index, it also must be that

$$\begin{aligned} |-1 - L| &< \frac{1}{3} \\ -\frac{1}{3} &< L + 1 < \frac{1}{3} \\ -\frac{4}{3} &< L < -\frac{2}{3} \end{aligned}$$

There is no number L with $\frac{2}{3} < L < \frac{4}{3}$ and $-\frac{4}{3} < L < -\frac{2}{3}$, so the sequence diverges.

Example 5. The sequence $\{\sqrt{n}\}$ diverges differently.

Definition: The sequence $\{a_n\}$ **diverges to infinity** if for every real number M there exists an integer N such that for all n,

$$\text{if } n > N \text{ then } a_n > M.$$

We denote this as $\lim\limits_{n\to\infty} a_n = \infty$ or $a_n \to \infty$.

Similarly, the sequence $\{a_n\}$ **diverges to negative infinity** if for every real number m there exists an integer N such that for all n,

$$\text{if } n > N \text{ then } a_n < m.$$

We denote this as $\lim\limits_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$.

The sequence $\{\sqrt{n}\}$ diverges to infinity. The sequence $\{1, 0, 2, 0, 3, 0, ...\}$ diverges, but does not diverge to infinity or negative infinity.

Sample Problems

1. Recall that if $a_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} a_n = 0$.
   a) Find a value of N for $\varepsilon = 0.15$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.15$
   b) Find a value of N for $\varepsilon = 0.01$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.01$
2. Define $a_n = \dfrac{3n+2}{5n+1}$.
   a) Find $\lim\limits_{n\to\infty} a_n$.
   b) Find a value of N for $\varepsilon = \dfrac{1}{2}$. That is, find a value for N so that for all $n > N$, $|a_n - L| < \dfrac{1}{2}$.
   c) Find a value of N for $\varepsilon = 0.05$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.05$.
   d) Find a value of N for $\varepsilon = 0.001$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.001$.
   e) Find a general expression for N in terms of ε.
3. Find the limit of the sequence $a_n = \dfrac{2n-5}{n+3}$ and use an epsilon-N proof to justify your answer.

Sample Problems- Solutions

1. Recall that if $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 0$.

a) Find a value of N for $\varepsilon = 0.15$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.15$

Solution: Let us start with what we need and solve for what that means for n. We want:

$$\begin{aligned} |a_n - 0| &< 0.15 \\ -0.15 &< a_n < 0.15 \\ -0.15 &< \frac{1}{n} < 0.15 \end{aligned}$$

The left-hand side is automatically true for all n since n is positive, so we just need to focus on the right-hand side.

$$\frac{1}{n} < 0.15$$

Both sides are positive and so we may take the reciprocal of both sides. Remember to reverse the inequality sign.

$$n > \frac{1}{0.15} = 6.\overline{6}$$

So $N = 6$ will work. Indeed, if $n > 6$, then

$$\frac{1}{n} \leq \frac{1}{7} \approx 0.142\,857 < 0.15$$

and then we also have

$$\begin{aligned} -0.15 &< \frac{1}{n} < 0.15 \\ -0.15 &< \frac{1}{n} - 0 < 0.15 \quad \text{same as} \quad \left|\frac{1}{n} - 0\right| < 0.15 \\ -0.15 &< a_n - 0 < 0.15 \quad \text{same as} \quad |a_n - 0| < 0.15 \end{aligned}$$

b) Find a value of N for $\varepsilon = 0.01$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.01$

Solution: Since $0.01 = \frac{1}{100}$, we can easily guess that $N = 100$ will do fine. Indeed, if $n > 100$, then

$$n > 100 \text{ and both sides being positive implies that } \frac{1}{n} < \frac{1}{100}$$

and so

$$\begin{aligned} -0.01 &< \frac{1}{n} < 0.01 \\ -0.01 &< \frac{1}{n} - 0 < 0.01 \quad \text{same as} \quad \left|\frac{1}{n} - 0\right| < 0.01 \quad \text{same as} \quad |a_n - 0| < 0.01 \end{aligned}$$

2. Define $a_n = \frac{3n+2}{5n+1}$.

a) Find $\lim_{n\to\infty} a_n$.

Solution: $a_n = \dfrac{3n\left(1 + \frac{2}{3n}\right)}{5n\left(1 + \frac{1}{5n}\right)}$ As n becomes larger and larger, $1 + \frac{2}{3n}$ and $1 + \frac{1}{5n}$ both approach 1 and so the sequence approaches $\frac{3}{5}$.

b) Find a value of N for $\varepsilon = \frac{1}{2}$. That is, find a value for N so that for all $n > N$, $|a_n - L| < \frac{1}{2}$.
Let us look at what we need. We need to find N so that if $n > N$, then

$$|a_n - L| < \varepsilon \quad \text{in this case this means that} \quad \left| \frac{3n+2}{5n+1} - \frac{3}{5} \right| < \frac{1}{2}$$

Let us simplify the expression $|a_n - L| = \left| \frac{3n+2}{5n+1} - \frac{3}{5} \right|$.

$$\left| \frac{3n+2}{5n+1} - \frac{3}{5} \right| = \left| \frac{5(3n+2)}{5(5n+1)} - \frac{3(5n+1)}{5(5n+1)} \right| = \left| \frac{15n+10-15n-3}{5(5n+1)} \right| = \left| \frac{7}{25n+5} \right| = \frac{7}{25n+5}$$

$$\left| \frac{3n+2}{5n+1} - \frac{3}{5} \right| < \frac{1}{2} \quad \text{is the same as} \quad \frac{7}{25n+5} < \frac{1}{2}$$

We solve this inequality:

$$\begin{aligned} \frac{7}{25n+5} &< \frac{1}{2} && 2(25n+5) \text{ is positive} \\ 14 &< 25n+5 && \text{subtract } 5 \\ 9 &< 25n && \text{divide by } 25 \\ \frac{9}{25} &< n \end{aligned}$$

So all values of n, greater than $\frac{9}{25} = 0.36$ will work. So $N = 1$

c) Find a value of N for $\varepsilon = 0.05$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.05$.
Let us look at what we need. We need to find N so that if $n > N$, then

$$|a_n - L| < \varepsilon \quad \text{in this case this means that} \quad \left| \frac{3n+2}{5n+1} - \frac{3}{5} \right| < 0.05$$

We already saw that $\left| \frac{3n+2}{5n+1} - \frac{3}{5} \right| = \frac{7}{25n+5}$, so we need that

$$\frac{7}{25n+5} < 0.05$$

We solve this inequality:

$$\begin{aligned} \frac{7}{25n+5} &< 0.05 && \text{both sides are positive - take reciprocal} \\ \frac{25n+5}{7} &> \frac{1}{0.05} && \frac{1}{0.05} = 20 \\ \frac{25n+5}{7} &> 20 && \text{multiply by } 7 \\ 25n+5 &> 140 && \text{subtract } 5 \\ 25n &> 135 && \text{divide by } 25 \\ n &> \frac{135}{25} = 5.4 \end{aligned}$$

So all values of n, greater than 5.4 will work. So $N = 5$.

d) Find a value of N for $\varepsilon = 0.001$. That is, find a value for N so that for all $n > N$, $|a_n - L| < 0.001$. Let us look at what we need. We need to find N so that if $n > N$, then

$$|a_n - L| < \varepsilon \text{ in this case this means that } \left|\frac{3n+2}{5n+1} - \frac{3}{5}\right| < 0.001$$

we have shown already that $\left|\frac{3n+2}{5n+1} - \frac{3}{5}\right| = \frac{7}{25n+5}$

$$\left|\frac{3n+2}{5n+1} - \frac{3}{5}\right| < 0.001 \quad \text{same as } \frac{7}{25n+5} < 0.001$$

We solve this inequality:

$$\begin{aligned}
\frac{7}{25n+5} &< 0.001 && \text{both sides are positive- take reciprocal} \\
\frac{25n+5}{7} &> \frac{1}{0.001} && \frac{1}{0.001} = 1000 \\
\frac{25n+5}{7} &> 1000 && \text{multiply by 7} \\
25n+5 &> 7000 && \text{subtract 5} \\
25n &> 6995 && \text{divide by 5} \\
n &> \frac{6995}{25} = 279.8
\end{aligned}$$

So all values of n, greater than 279.8 will work. So $N = 279$.

e) Find a general expression for N in terms of ε.

Solution: $|a_n - L| < \varepsilon$ in this case this means that $\left|\frac{3n+2}{5n+1} - \frac{3}{5}\right| < \varepsilon$. The expression $\left|\frac{3n+2}{5n+1} - \frac{3}{5}\right|$ can be simplified as $\frac{7}{25n+5}$.

$$\begin{aligned}
|a_n - L| &< \varepsilon \\
\frac{7}{25n+5} &< \varepsilon && \text{both sides are positive - take reciprocal} \\
\frac{25n+5}{7} &> \frac{1}{\varepsilon} && \text{multiply by 7} \\
25n+5 &> \frac{7}{\varepsilon} && \text{subtract 5} \\
25n &> \frac{7}{\varepsilon} - 5 && \text{divide by 25} \\
n &> \frac{\frac{7}{\varepsilon} - 5}{25} = \frac{7}{25\varepsilon} - \frac{5}{25} = \frac{7}{25\varepsilon} - \frac{1}{5}
\end{aligned}$$

So $N = \lceil \frac{7}{25\varepsilon} - \frac{1}{5} \rceil$ is a good value for N. However, once we find a value for N that works, any greater value would also work. So, we migh present a simpler value for N, namely $\lceil \frac{7}{25\varepsilon} \rceil$. As long as we are "growing" this value, we are OK.

Let us prove that $N = \lceil \frac{7}{25\varepsilon} \rceil$ indeed works.

If $n > N$, then $n > \dfrac{7}{25\varepsilon}$. Then $n > \dfrac{7}{25\varepsilon} - \dfrac{1}{5}$ is also true.

$$\begin{array}{rcll}
n & > & \dfrac{7}{25\varepsilon} - \dfrac{1}{5} & \text{add } \dfrac{1}{5} \\
n + \dfrac{1}{5} & > & \dfrac{7}{25\varepsilon} & \\
\dfrac{5n+1}{5} & > & \dfrac{7}{25\varepsilon} & \text{both sides are positive - take reciprocal} \\
\dfrac{5}{5n+1} & < & \dfrac{25\varepsilon}{7} & \text{multiply by } \dfrac{7}{25} \\
\dfrac{7 \cdot 5}{25(5n+1)} & < & \varepsilon & \\
\dfrac{7}{5(5n+1)} & < & \varepsilon &
\end{array}$$

and we already know that $\dfrac{7}{5(5n+1)}$ is $\left| a_n - \dfrac{3}{5} \right|$ and so we have that $\left| a_n - \dfrac{3}{5} \right| < \varepsilon$. This completes our proof.

3. Find the limit of the sequence $a_n = \dfrac{2n-5}{n+3}$ and use an epsilon-N proof to justify your answer.

Solution: As n approaches infinity, $a_n = \dfrac{2n\left(1 - \dfrac{5}{2n}\right)}{n\left(1 + \dfrac{3}{n}\right)}$ will approach 2. Let $\varepsilon > 0$ be given.

Part 1 - This is how we find a correct value of N. You do not need to present this.

$$|a_n - L| < \varepsilon \quad \text{means that} \quad \left| \frac{2n-5}{n+3} - 2 \right| < \varepsilon$$

We simplify $\left| \dfrac{2n-5}{n+3} - 2 \right|$:

$$\left| \frac{2n-5}{n+3} - 2 \right| = \left| \frac{2n-5}{n+3} - \frac{2(n+3)}{n+3} \right| = \left| \frac{2n-5-2n-6}{n+3} \right| = \left| \frac{-11}{n+3} \right| = \frac{11}{n+3}$$

So we need that

$$\begin{array}{rcll}
\dfrac{11}{n+3} & < & \varepsilon & \text{both sides are positive - take reciprocal} \\
\dfrac{n+3}{11} & > & \dfrac{1}{\varepsilon} & \text{multiply by } 11 \\
n+3 & > & \dfrac{11}{\varepsilon} & \text{subtract } 3 \\
n & > & \dfrac{11}{\varepsilon} - 3 &
\end{array}$$

So $N = \left\lceil \dfrac{11}{\varepsilon} - 3 \right\rceil$ is good, and so is any greater N, for example $\left\lceil \dfrac{11}{\varepsilon} \right\rceil$

Part 2 - This is how we present a correct value of N.

Claim: $\lim_{n\to\infty} \dfrac{2n-5}{n+3} = 2$

proof: Let $\varepsilon > 0$ be given. Define $N = \lceil \dfrac{11}{\varepsilon} \rceil$. If $n > N$, then

$$\begin{aligned} n &> \frac{11}{\varepsilon} - 3 && \text{add 3} \\ n+3 &> \frac{11}{\varepsilon} && \text{both sides are positive - take reciprocal} \\ \frac{1}{n+3} &< \frac{\varepsilon}{11} && \text{multiply by 11} \\ \frac{11}{n+3} &< \varepsilon \end{aligned}$$

Also,

$$|a_n - L| = \left| \frac{2n-5}{n+3} - 2 \right| = \left| \frac{2n-5-2n-6}{n+3} \right| = \left| \frac{-11}{n+3} \right| = \frac{11}{n+3} < \varepsilon$$

This completes our proof.

