The real numbers has the **completeness property**: If a non-empty set of real numbers is bounded above, then there exists a least upper bound; if a non-empty set of real numbers is bounded below, then there exists a greatest lower bound.

Definition: The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there exists an integer N such that for all n,

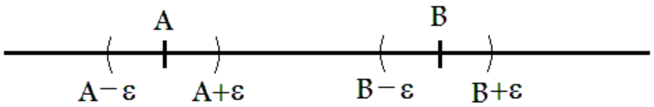
$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon.$$

If no such number L exists, we say $\{a_n\}$ **diverges**.
If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ and call L the **limit** of the sequence.

Notice that $|a_n - L| < \varepsilon$ and $L - \varepsilon < a_n < L + \varepsilon$ are equivalent statements.

Theorem 1. Convergent sequences have unique limits: if $\{a_n\}$ is a sequence with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} a_n = B$, then $A = B$.

Proof: Suppose for a contradiction that a sequence a_n converges to two different numbers A and B. The basic idea here is that if ε is selected to be small enough, then the ε neighborhood of A will be disjoint of the ε neighborhood of B and so a_n can not be in both intervals.



Suppose that $A \neq B$. We may assume that $A < B$. (Otherwise just re-label them so that the larger number is denoted by B.) Define $\varepsilon = \dfrac{B-A}{2}$. Since $\{a_n\}$ converges to A, there exists N_A so that for all $n > N_A$,

$$A - \varepsilon < a_n < A + \varepsilon$$

Similarly, since $\{a_n\}$ converges to B, there exists N_B so that for all $n > N_B$,

$$B - \varepsilon < a_n < B + \varepsilon$$

Now let $n > \max(N_A, N_B)$, so both conditions hold. Then

$$A - \varepsilon < a_n < A + \varepsilon \quad \text{and} \quad B - \varepsilon < a_n < B + \varepsilon$$

We will only need the right-hand side of the first inequality and the left-hand side of the other:

$$\begin{aligned}
a_n &< A + \varepsilon \quad \text{and} \quad B - \varepsilon < a_n \qquad \text{recall that } \varepsilon = \frac{B-A}{2} \\
a_n &< A + \frac{B-A}{2} \quad \text{and} \quad B - \frac{B-A}{2} < a_n \\
a_n &< \frac{2A}{2} + \frac{B-A}{2} \quad \text{and} \quad \frac{2B}{2} - \frac{B-A}{2} < a_n \\
a_n &< \frac{2A+B-A}{2} \quad \text{and} \quad \frac{2B-B+A}{2} < a_n \\
a_n &< \frac{A+B}{2} \quad \text{and} \quad \frac{A+B}{2} < a_n
\end{aligned}$$

These two can not be true at the same time. This is a contradiction, so $A \neq B$ is impossible. This completes our proof.

In the following, we will prove properties of limits that enable us to compute limits based on other limits.

Theorem 2. (Sum Rule) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that A and B are real numbers such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then $\lim_{n\to\infty} (a_n + b_n) = A + B$.

Proof: Suppose that A and B are real numbers such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Let $\varepsilon > 0$ be given. There exist N_a and N_b natural numbers such that for all $n > N_a$,

$$A - \frac{\varepsilon}{2} < a_n < A + \frac{\varepsilon}{2}$$

and for all $k > N_b$,

$$B - \frac{\varepsilon}{2} < b_k < B + \frac{\varepsilon}{2}$$

Let $N = \max(N_a, N_b)$. If $n > N$, then

$$A - \frac{\varepsilon}{2} < a_n < A + \frac{\varepsilon}{2} \text{ and } B - \frac{\varepsilon}{2} < b_n < B + \frac{\varepsilon}{2}$$

Adding these two inequalities we obtain

$$\begin{aligned} A - \frac{\varepsilon}{2} + B - \frac{\varepsilon}{2} &< a_n + b_n < A + \frac{\varepsilon}{2} + B + \frac{\varepsilon}{2} \\ A + B - \varepsilon &< a_n + b_n < A + B + \varepsilon \end{aligned}$$

Thus $a_n + b_n$ converges to $A + B$.

Example 1.

a) $\lim_{n\to\infty} \left(2 + \frac{1}{n}\right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n} = 2 + 0 = 2$

b) $\lim_{n\to\infty} \frac{3n+1}{n} = \lim_{n\to\infty} \left(\frac{3n}{n} + \frac{1}{n}\right) = \lim_{n\to\infty} \left(3 + \frac{1}{n}\right) = \lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n} = 3 + 0 = 3$

Theorem 3. (Difference Rule) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that A and B are real numbers such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then $\lim_{n\to\infty} (a_n - b_n) = A - B$.

Proof: Suppose that A and B are real numbers such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Let $\varepsilon > 0$ be given. There exist N_a and N_b natural numbers such that for all $n > N_a$,

$$A - \frac{\varepsilon}{2} < a_n < A + \frac{\varepsilon}{2}$$

and for all $k > N_b$,

$$B - \frac{\varepsilon}{2} < b_k < B + \frac{\varepsilon}{2}$$

Multiply all sides by -1.

$$-B + \frac{\varepsilon}{2} > -b_k > -B - \frac{\varepsilon}{2}$$

We turn the inequality around:

$$-B - \frac{\varepsilon}{2} < -b_k < -B + \frac{\varepsilon}{2}$$

Let $N = \max(N_a, N_b)$. If $n > N$, then

$$A - \frac{\varepsilon}{2} < a_n < A + \frac{\varepsilon}{2} \text{ and } \quad -B - \frac{\varepsilon}{2} < -b_n < -B + \frac{\varepsilon}{2}$$

Adding these two inequalities we obtain

$$\begin{aligned} A - \frac{\varepsilon}{2} + (-B) - \frac{\varepsilon}{2} &< a_n + (-b_n) < A + \frac{\varepsilon}{2} + (-B) + \frac{\varepsilon}{2} \\ A - B - \varepsilon &< a_n - b_n < A - B + \varepsilon \end{aligned}$$

Thus $a_n - b_n$ converges to $A - B$.

Theorem 4. (Constant Multiple Rule) Let $\{a_n\}$ be a sequence of real numbers and c a real number. Suppose that $\lim\limits_{n\to\infty} a_n = A$. Then $\lim\limits_{n\to\infty} (ca_n) = cA$.

Proof: Case 1. Suppose that $c = 0$. Then ca_n is a constant sequence and its limit is clearly zero.

Case 2. Suppose that $c \neq 0$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = A$, there exists $N > 0$ so that for all $n > N$,

$$\begin{aligned} |a_n - A| &< \frac{\varepsilon}{|c|} \qquad \text{mulitply by } |c| \\ |c|\,|a_n - A| &< \varepsilon \\ |ca_n - cA| &< \varepsilon \end{aligned}$$

and so $\lim\limits_{n\to\infty} ca_n = cA$.

Theorem 5. (Product Rule) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that A and B are real numbers such that $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$. Then $\lim\limits_{n\to\infty} (a_n b_n) = AB$

Theorem 6. (Quotient Rule) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that A and $B \neq 0$ are real numbers such that $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$. Then $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$.

Proving these theorems is more difficult. We will not cover it in this course.

Example 2.

a) $\lim\limits_{n\to\infty} \left(-\frac{3}{n^2}\right) = -3 \lim\limits_{n\to\infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right) = -3 \lim\limits_{n\to\infty} \frac{1}{n} \cdot \lim\limits_{n\to\infty} \frac{1}{n} = -3 \cdot 0 \cdot 0 = 0$

b) $\lim\limits_{n\to\infty} \frac{3 - 2n^4}{7n^4 + 2} = \lim\limits_{n\to\infty} \frac{\frac{3}{n^4} - 2}{7 + \frac{2}{n^4}} = \frac{\lim\limits_{n\to\infty} \left(\frac{3}{n^4} - 2\right)}{\lim\limits_{n\to\infty} \left(7 + \frac{2}{n^4}\right)} = \frac{-2}{7} = -\frac{2}{7}$

Theorem 7. (The Sandwich Theorem for Sequences): Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$. Suppose that there exists N positive integer such that for all $n > N$,

$$a_n \leq b_n \leq c_n$$

then b_n converges to L.

Proof: Suppose the conditions hold. Let $\varepsilon > 0$ be given. Since $\{a_n\}$ and $\{b_n\}$ are converge to L, there exist N_a and N_c such that when $n > N_a$, then

$$L - \varepsilon < a_n < L + \varepsilon$$

and when $n > N_c$, then

$$L - \varepsilon < c_n < L + \varepsilon$$

Let $N = \max(N_a, N_b)$. If $n > N$, then

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

and so $\{b_n\}$ converges to L.

Consequence: If $|b_n| \leq c_n$ and $c_n \to 0$, then $b_n \to 0$.

Example 3.

a) $\dfrac{\sin n}{n} \to 0$ since $-\dfrac{1}{n} \leq \dfrac{\sin n}{n} \leq \dfrac{1}{n}$ and $\dfrac{1}{n} \to 0$ and $-\dfrac{1}{n} \to 0$

b) $\dfrac{(-1)^n}{n^2} \to 0$ since $-\dfrac{1}{n^2} \leq \dfrac{(-1)^n}{n^2} \leq \dfrac{1}{n^2}$

Some sequences are defined **recursively**. Recursive definitions enable us to compute the first, second, third, ... nth term, but we cannot compute the nth term without first computing the first $n-1$.
The **Fibonacci sequence** is a perfect example for this.

Definition: The **Fibonacci sequence** is defined recursively as

$$F_1 = 1, \quad F_2 = 1, \quad \text{and for all } n \in \mathbb{N}, \quad F_{n+2} = F_n + F_{n+1}$$

The first few terms of the Fibonacci Sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$ The explicit formula for the nth term of this sequence is a very interesting formula.

Definition: A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** for the sequence $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is called a **lower bound** for the sequence $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and from below, we say that $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, we say that $\{a_n\}$ is an **unbounded** sequence.

Example 4. a) The sequence $1, 4, 9, 16, ...$ is bounded from below but not from above. 1 is the greatest lower bound for the sequence.

b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ..., \frac{n}{n+1}$ is bounded. $\frac{1}{2}$ is the greatest lower bound and 1 is the lowest upper bound for this sequence.

Theorem 8. If $\{a_n\}$ is convergent, then a_n is bounded.

proof: Suppose that $\{a_n\}$ is a convergent sequence and $a_n \to L$. Let $\varepsilon = 1$. There exists a natural number N such that for all $n > N$,

$$L - 1 < a_n < L + 1$$

Consider now the set $\{a_1, a_2, a_3, ..., a_N\}$. Since this is a finite set, it has a lowest and greatest element. Denote these by m and M, respectively. We claim that $\min(m, L-1)$ is a lower bound for the sequence $\{a_n\}$ and $\max(M, L+1)$ is an upper bound for the sequence.

Let a_k be any term of the sequence. If $k > N$, then $L - 1 < a_k < L + 1$ and so

$$\min(m, L-1) \leq L - 1 < a_k < L + 1 \leq \max(M, L+1)$$

and if $k \leq N$, then a_k is in the set $\{a_1, a_2, a_3, ..., a_N\}$ and so

$$\min(m, L-1) \leq m < a_k < M \leq \max(M, L+1)$$

Defintion: A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n. That is, $a_1 \leq a_2 \leq a_3 \leq ...$ The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n. That is, $a_1 \geq a_2 \geq a_3 \geq ...$ The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

Example 5. a) The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ is nonincreasing.

b) the constant sequence $2, 2, 2, ...$ is both nonincreasing and nondecreasing.

c) the sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, ...$ is not monotonic.

Theorem 9. If a sequence $\{a_n\}$ is bounded from above and non-decreasing, then it is also convergent. (Similarly, if a sequence is bounded from below and non-increasing, then it is convergent.)

proof: Suppose that $\{a_n\}$ is bounded and nondecreasing. Let L be the least upper bound for the sequence. Since L is an upper bound, $a_n \leq L$ for all n.

Let $\varepsilon > 0$ be given. Since L is the lowest upper bound, $L - \varepsilon$ is NOT an upper bound. This means that there exists m natural number such that $a_m > L - \varepsilon$. Since a_n is nondecreasing, all subsequent terms will have this property, i.e. for all $n > m$, $a_n \geq a_m > L - \varepsilon$. thus we have that for all $n > m$

$$L - \varepsilon < a_n \leq L < L + \varepsilon$$

and so $L - \varepsilon < a_n < L + \varepsilon$ and so a_n converges to L. The proof for nonincreasing sequences is similar.