Sequences Defined Recursively

Definition: The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there exists an integer N such that for all n,

$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon.$$

If no such number L exists, we say $\{a_n\}$ **diverges**.
If $\{a_n\}$ converges to L, we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ and call L the **limit** of the sequence.

Some sequences are defined **recursively**. Recursive definitions enable us to compute the first, second, third, ... nth term, but we cannot compute the nth term without first computing the first $n-1$.
The **Fibonacci sequence** is a perfect example for this.

Definition: The **Fibonacci sequence** is defined recursively as

$$F_1 = 1, \quad F_2 = 1, \quad \text{and for all } n \in \mathbb{N}, \quad F_{n+2} = F_n + F_{n+1}$$

The first few terms of the Fibonacci Sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$ The explicit formula for the nth term of this sequence is a very interesting formula.

The following method can be used to find the limit of a recursive sequence - provided that we have already established that the limit exists.

Example 1. Consider the sequence defined recursively: $a_1 = 2$ and $a_{n+1} = \frac{1}{2}a_n + 3$. Assume that this limit exists and find its value.

Solution: This method is based on the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$. Let us denote this limit by x.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\ \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\frac{1}{2}a_n + 3\right) \end{aligned}$$

Using properties of limits, we re-write the right-hand side:

$$\lim_{n\to\infty} \left(\frac{1}{2}a_n + 3\right) = \lim_{n\to\infty} \left(\frac{1}{2}a_n\right) + \lim_{n\to\infty} 3 = \frac{1}{2}\lim_{n\to\infty} a_n + 3$$

So now we have:

$$\begin{aligned} \lim_{n\to\infty} a_n &= \frac{1}{2}\lim_{n\to\infty} a_n + 3 & & \\ x &= \frac{1}{2}x + 3 & & \text{subtract } \frac{1}{2}x \\ \frac{1}{2}x &= 3 & & \text{multiply by } 2 \\ x &= 6 & & \end{aligned}$$

So this limit is 6.

Example 2. Consider the sequence defined recursively: $a_1 = 0$, $a_2 = 1$, and $a_{n+1} = a_n + 2a_{n-1}$. Assume that the limit $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists and find its value.

Solution: This method is based on the fact that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{a_n}{a_{n-1}}$. Let us denote this limit by x.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a_n + 2a_{n-1}}{a_n} = \lim_{n\to\infty} \frac{a_n}{a_n} + \lim_{n\to\infty} \frac{2a_{n-1}}{a_n} = 1 + 2\lim_{n\to\infty} \frac{a_{n-1}}{a_n}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= 1 + 2\lim_{n\to\infty} \frac{a_{n-1}}{a_n} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= 1 + \frac{2}{\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n-1}}} \\
x &= 1 + \frac{2}{x} && \text{multiply by } x \\
x^2 &= x + 2 \\
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0 \qquad \Longrightarrow \qquad x_1 = 2 \quad x_2 = -1
\end{aligned}$$

since all terms of this sequence are positive, the limit is the positive solution, $x = 2$.

Example 3. Consider the sequence $a_1 = 1$ and $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$. Assume that the limit $\lim\limits_{n\to\infty} a_n$ exists and find its value.

Solution: This method is based on the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$. Let us denote this limit by x.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\frac{1}{2}\left(a_n + \frac{2}{a_n}\right)\right) \\
\lim_{n\to\infty} a_n &= \frac{1}{2}\lim_{n\to\infty} \left(a_n + \frac{2}{a_n}\right) \\
\lim_{n\to\infty} a_n &= \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{2}{\lim\limits_{n\to\infty} a_n}\right) \\
x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\
x &= \frac{1}{2}x + \frac{1}{x} && \text{subtract } \frac{1}{2}x \\
\frac{1}{2}x &= \frac{1}{x} && \text{multiply by } x \\
\frac{1}{2}x^2 &= 1 && \text{multiply by } 2 \\
x^2 &= 2 \\
x &= \pm\sqrt{2}
\end{aligned}$$

Since all terms of the sequence are positive, the negative solution is ruled out and the limit must be $\sqrt{2}$.

Example 4. Consider the sequence $a_1 = 5$ and $a_{n+1} = \dfrac{a_n + 3}{a_n + 1}$. Assume that the limit $\lim\limits_{n\to\infty} a_n$ exists and find its value.

Solution: This method is based on the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$. Let us denote this limit by x.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{a_n + 3}{a_n + 1} \\
\lim_{n\to\infty} a_n &= \frac{\lim\limits_{n\to\infty} a_n + 3}{\lim\limits_{n\to\infty} a_n + 1} \\
x &= \frac{x+3}{x+1} \qquad \text{multiply by } x+1 \\
x(x+1) &= x+3 \\
x^2 + x &= x + 3 \\
x^2 &= 3 \\
x &= \pm\sqrt{3}
\end{aligned}$$

Since all terms of the sequence are positive, the negative solution is ruled out and the limit must be $\sqrt{3}$.

Example 5. Compute the value of $\sqrt{20 + \sqrt{20 + \sqrt{20 + \ldots}}}$

Solution: Define $a_1 = 20$ and $a_{n+1} = \sqrt{20 + a_n}$. Assuming the limit exists, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$. Let us denote this limit by x.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{20 + a_n}
\end{aligned}$$

We will use a theorem here we did not yet prove: that $\lim\limits_{n\to\infty} \sqrt{c_n} = \sqrt{\lim\limits_{n\to\infty} c_n}$. Let us assume this property for now. Using properties of limits,

$$\lim_{n\to\infty} \sqrt{20 + a_n} = \sqrt{\lim_{n\to\infty} (20 + a_n)} = \sqrt{\lim_{n\to\infty} 20 + \lim_{n\to\infty} a_n} = \sqrt{20 + \lim_{n\to\infty} a_n}$$

So now we have that

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \sqrt{20 + \lim_{n\to\infty} a_n} \\
x &= \sqrt{20 + x} \\
x^2 &= x + 20 \\
x^2 - x - 20 &= 0 \\
(x-5)(x+4) &= 0 \qquad \Longrightarrow \qquad x_1 = 5 \quad x_2 = -4
\end{aligned}$$

Since all terms of the sequence are positive, the negative solution is ruled out and the limit must be 5.

Example 6. Let $F_1, F_2, F_3, \ldots$ be the Fibonacci sequence defined by $F_1 = 1$, $F_2 = 1$, and for $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$. Compute $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$.

Solution: Let us denote $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ by x. We will state that $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}}$ and solve for x.

$$\begin{aligned}
\lim_{n\to\infty} \frac{F_{n+1}}{F_n} &= \lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}} \\
\lim_{n\to\infty} \frac{F_{n+1}}{F_n} &= \lim_{n\to\infty} \frac{F_n + F_{n+1}}{F_{n+1}} \\
\lim_{n\to\infty} \frac{F_{n+1}}{F_n} &= \lim_{n\to\infty} \left(\frac{F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+1}} \right) \\
\lim_{n\to\infty} \frac{F_{n+1}}{F_n} &= \lim_{n\to\infty} \frac{F_n}{F_{n+1}} + 1 \\
x &= \frac{1}{x} + 1 \\
x^2 &= 1 + x \\
x^2 - x - 1 &= 0
\end{aligned}$$

$$x_{1,2} = \frac{1 \pm \sqrt{1-(-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since all terms of the sequence are positive, the negative solution, $\dfrac{1-\sqrt{5}}{2}$ is ruled out and so the limit must be $\dfrac{1+\sqrt{5}}{2}$.

Practice Problems

Assume that the following sequences all have limits. In case of each of the sequences given, find the value of its limit.

1. $a_1 = 2$, $a_{n+1} = \dfrac{12}{1+a_n}$
2. $a_1 = -1$, $a_{n+1} = \dfrac{a_n + 25}{a_n + 1}$
3. $a_1 = -1$, $a_{n+1} = \sqrt{5 + 2a_n}$
4. $a_1 = 2$, $a_{n+1} = \sqrt{2a_n}$
5. $a_1 = 3$, $a_{n+1} = 6 - \sqrt{a_n}$
6. $a_1 = 10$, $a_{n+1} = 4 + \dfrac{21}{a_n}$
7. $a_1 = 2$, $a_{n+1} = \dfrac{1}{2+a_n}$
8. $a_1 = 1$, $a_{n+1} = \sqrt{1+a_n}$
9. Consider the sequence defined recursively as $a_1 = 1$, $a_2 = 1$, $a_{n+2} = 2a_n + a_{n+1}$. Compute $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.
10. Compute the value of the infinite continued fraction $\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}$
11. A farmer plants A acres of wheat one year. Each year thereafter, he harvests (removes) $\dfrac{1}{4}$ of the planted acreage and then plants 1500 more acres. The number of acres of wheat planted approaches what number?

Answers - Practice Problems

1.) 3 2.) 5 3.) $\sqrt{6}+1$ 4.) 2 5.) 4 6.) 7 7.) $\sqrt{2}-1$ 8.) $\dfrac{\sqrt{5}+1}{2}$

9.) 2 10.) $\sqrt{2}-1$ 11.) 6000 acres