Theorem: (The Continuous Function Theorem for Sequences) Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example 1. a) $\sqrt{\dfrac{n+1}{n}} \to 1$

proof: $\dfrac{n+1}{n} \to 1$ and $f(x) = \sqrt{x}$ is continuous at $x = 1$. Thus $\sqrt{\dfrac{n+1}{n}} \to \sqrt{1} = 1$

b) $\ln\left(\dfrac{n^2-1}{n^2+1}\right) \to \ln 1 = 0$

c) $2^{1/n} \to 2^0 = 1$

Theorem: Suppose that $f(x)$ is a function defined for all $x \geq n_0$ for some $n_0 \in \mathbb{N}$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for all $n \geq n_0$ Then

$$\text{if } \lim_{x\to\infty} f(x) = L, \text{ then } \lim_{n\to\infty} a_n = L$$

Then we can apply L'Hôpital's rule to find limits of sequences.

Example 2. a) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$

proof: $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

b) $\displaystyle\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = e^2$

proof: $\ln a_n = \ln\left(\dfrac{n+1}{n-1}\right)^n = n \ln\left(\dfrac{n+1}{n-1}\right) = \dfrac{\ln\left(\dfrac{n+1}{n-1}\right)}{\dfrac{1}{n}}$ - This is an indeterminate of type $\dfrac{0}{0}$.

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln\left(\dfrac{n+1}{n-1}\right)}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{-\dfrac{2}{n^2-1}}{-\dfrac{1}{n^2}} = \lim_{n\to\infty} \left(-\frac{2}{n^2-1}\right)\left(-n^2\right) = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2$$

$\ln a_n \to 2$ and $f(x) = e^x$ is continuous on $\mathbb{R}$. thus $a_n = e^{\ln a_n} \to e^2$

Commonly Occurring Limits. In each of the following, x is a fixed number.

1. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\displaystyle\lim_{n\to\infty} x^{1/n} = 1 \qquad x > 0$
4. $\displaystyle\lim_{n\to\infty} x^n = 0 \qquad -1 < x < 1$
5. $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

proof. 1) was done using L'Hôpital's rule.

2) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

proof: $\lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} n^{1/n} = \lim\limits_{x\to\infty} x^{1/x}$ We will think of $x^{1/x}$ as $e^{\ln\left(x^{1/x}\right)}$

$\lim\limits_{x\to\infty} \ln\left(x^{1/x}\right) = \lim\limits_{x\to\infty} \frac{1}{x}\ln x = \lim\limits_{x\to\infty} \frac{\ln x}{x} = 0$

$f(x) = e^x$ is continuous on $\mathbb{R}$

$\ln\left(x^{1/x}\right) \to 0$ and so $e^{\ln\left(x^{1/x}\right)} \to e^0$ and so $x^{1/x} \to 1$

3) $\lim\limits_{n\to\infty} a^{1/n} = 1 \qquad a > 0$

proof: We will prove that $\lim\limits_{x\to\infty} a^{1/x} = 1$

$$\lim_{x\to\infty} a^{1/x} = \lim_{x\to\infty} e^{\ln a^{1/x}} = \lim_{x\to\infty} e^{1/x \ln a} = e^0 = 1$$

4) $\lim\limits_{n\to\infty} x^n = 0$ for all x with $-1 < x < 1$

proof: Let $\varepsilon > 0$ be given. Since (by limit 3) $\lim\limits_{n\to\infty} \varepsilon^{1/n} = 1$ and $|x| < 1$, there exists N such that for all $n > N$,

$$\varepsilon^{1/n} > |x| \quad \text{and so} \quad \varepsilon > |x|^n \geq x^n$$

5.) $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any x

proof: We will present two methods. In the first method, we will use the fact that $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{\frac{n}{x}}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}\cdot x} = \lim_{n\to\infty} \left[\left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x = \left[\lim_{n\to\infty} \left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x = e^x$$

Method 2: We will use the identity $x = e^{\ln x}$ and L'Hôpital's rule.

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n} = e^{\lim\limits_{n\to\infty} \ln\left(1+\frac{x}{n}\right)^n} = e^{\lim\limits_{n\to\infty} n\ln\left(1+\frac{x}{n}\right)} = e^{\lim\limits_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}} = e^M$$

The exponent is now an indeterminate of the type $\frac{0}{0}$, so we may apply L'Hôpital's rule. We differentiate with respect to n.

$$M = \lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{\frac{n+x}{n}}\cdot\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{n+x}\cdot\left(-\frac{x}{n^2}\right)\cdot\left(-n^2\right) = \lim_{n\to\infty} \frac{nx}{n+x}$$

We may apply L'Hôpital's rule again or just divide both numerator and denominator by n to see that this limit $M = x$:

$$\lim_{n\to\infty} \frac{nx}{n+x} = \lim_{n\to\infty} \frac{x}{1+\frac{x}{n}} = x$$

and so the entire limit is $e^M = e^x$.

6.) $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0 \qquad$ (any x)

proof: Let us first assume that $x > 0$. Let M be an integer for which $\dfrac{x}{M} < 1$.

$$\begin{aligned}
\frac{x^n}{n!} &= \frac{x \cdot x \cdot \ldots \cdot x}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} = \frac{x \cdot x \cdot \ldots \cdot x}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot M \cdot (M+1) \cdot (M+2) \cdot \ldots \cdot n} \\
&\leq \frac{x^n}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot M \cdot M \cdot M \cdot \ldots \cdot M} = \frac{x^n}{M! \cdot M^{n-M}} \\
&= \frac{x^n}{M! \cdot M^{n-M}} \cdot \frac{M^M}{M^M} = \frac{M^M}{M!} \cdot \frac{x^n}{M^n} = \frac{M^M}{M!} \left(\frac{x}{M}\right)^n
\end{aligned}$$

Since $\dfrac{M^M}{M!}$ is a constant and $\left(\dfrac{x}{M}\right)^n \to 0$, so does $\dfrac{x^n}{n!}$.
If $x \leq 0$, then we use the sandwich theorem:

$$-\left|\frac{x^n}{n!}\right| \leq \frac{x^n}{n!} \leq \left|\frac{x^n}{n!}\right|$$

and since $\left|\dfrac{x^n}{n!}\right| \to 0$, we also have $\dfrac{x^n}{n!} \to 0$.

