1. In each case, decide whether the sequence is convergent or divergent. If convergent, find its limit.

a) $a_n = 5 - 0.1^n$

b) $a_n = \dfrac{2-n}{7+3n}$

c) $a_n = 1^n + (-1)^n$

d) $a_n = \sqrt{\dfrac{2n}{n+1}}$

e) $a_n = \dfrac{\sin n}{n}$

f) $a_n = \dfrac{3^{n+1}}{2^{2n+1}}$

g) $a_n = \dfrac{3^{n-1}}{2^{n+3}}$

h) $a_n = n\pi \cos\left(\dfrac{n\pi}{2}\right)$

i) $a_n = 8^{1/n}$

j) $a_n = \dfrac{\ln n}{\ln 2n}$

k) $a_n = \left(\dfrac{n+1}{2n}\right)\left(1-\dfrac{1}{n}\right)$

l) $a_n = \dfrac{n!}{10^n}$

m) $a_n = \dfrac{5^n}{n!}$

n) $a_n = (-1)^n \dfrac{\sin n}{n^2}$

o) $a_n = \lim\limits_{n\to\infty} \sqrt[n]{2n}$

2. Assume that each of the following sequences are convergent and find the value of the limit.

a) $a_1 = 8$ and $a_{n+1} = \sqrt{a_n + 6}$

b) $a_1 = 5$ and $a_{n+1} = \sqrt{2a_n - 1}$

c) $a_1 = \sqrt{30}$ and $a_{n+1} = \sqrt{30 + a_n}$

d) $a_1 = 2$ and $a_{n+1} = \dfrac{1}{2 + a_n}$

e) $a_1 = 1$ and $a_{n+1} = \dfrac{1}{3}(a_n + 12)$

f) $a_1 = 2$ and $a_{n+1} = \dfrac{a_n + 1}{2a_n + 1}$

g) $a_1 = \sqrt{5}$ and $a_{n+1} = \sqrt{5a_n}$

h) $a_1 = 1$ and $a_{n+1} = \dfrac{a_n^2 + 4}{a_n + 2}$

i) $a_1 = 10$ and $a_{n+1} = 0.6a_n + 8$

3. Assume that $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$ and $\lim\limits_{n\to-\infty}\left(1+\dfrac{1}{n}\right)^n = e$ and compute each of the following limits.

a) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+2}$

b) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{2n}$

c) $\lim\limits_{n\to\infty}\left(1+\dfrac{5}{n}\right)^{n}$

d) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n}$

Answers

1. a) converges to 5
   b) converges to $-\dfrac{1}{3}$
   c) diverges
   d) converges to $\sqrt{2}$
   e) converges to 0
   f) converges to 0
   g) diverges
   h) diverges
   i) converges to 1
   j) converges to 1
   k) converges to $\dfrac{1}{2}$
   l) diverges to infinity
   m) converges to 0
   n) converges to 1

2. a) 3 b) 1 c) 6 d) $\sqrt{2}-1$ e) 6 f) $\dfrac{\sqrt{2}}{2}$ g) 5 h) 2 i) 20

3. a) e b) e^2 c) e^5 d) $\dfrac{1}{e}$

Solutions

1. In each case, decide whether the sequence is convergent or divergent. If convergent, find its limit.

a) $a_n = 5 - 0.1^n$

Solution: We separate the two parts via the difference rule.

$$\lim_{n\to\infty} 5 - 0.1^n = \lim_{n\to\infty} 5 - \lim_{n\to\infty} 0.1^n$$

The limit of the constant sequence is that same value. Since $0.1 < 1$, 0.1^n approaches zero as n gets large.

$$\lim_{n\to\infty} 5 - 0.1^n = \lim_{n\to\infty} 5 - \lim_{n\to\infty} 0.1^n = 5 - 0 = 5$$

b) $a_n = \dfrac{2-n}{7+3n}$

Solution:

$$\lim_{n\to\infty} \frac{2-n}{7+3n} = \lim_{n\to\infty} \frac{\frac{2}{n} - 1}{\frac{7}{n} + 3} = \frac{\lim_{n\to\infty} \frac{2}{n} - \lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{7}{n} + \lim_{n\to\infty} 3} = \frac{0-1}{0+3} = -\frac{1}{3}$$

c) $a_n = 1^n + (-1)^n$

Solution: The first few terms of the sequence are: $2, 0, 2, 0, 2, 0, 2, 0, 2, 0 \ldots$ This sequence diverges.

d) $a_n = \sqrt{\dfrac{2n}{n+1}}$

Solution:

$$\lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} = \frac{2}{1+0} = 2$$

Since $f(x) = \sqrt{x}$ is continuous at $x = 2$,

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n+1}} = \sqrt{2}$$

e) $a_n = \dfrac{\sin n}{n}$

Solution: Tis sequence converges to zero by the sandwich rule.

$$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

Since $-1 \le \sin n \le 1$, $-\dfrac{1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$. Thus $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

f) $a_n = \dfrac{3^{n+1}}{2^{2n+1}}$

Solution:

$$\lim_{n\to\infty} \frac{3^{n+1}}{2^{2n+1}} = \lim_{n\to\infty} \frac{3\cdot 3^n}{2\cdot 2^{2n}} = \frac{3}{2}\lim_{n\to\infty} \frac{3^n}{\left(2^2\right)^n} = \frac{3}{2}\lim_{n\to\infty} \frac{3^n}{4^n} = \frac{3}{2}\lim_{n\to\infty} \left(\frac{3}{4}\right)^n$$

Since $\dfrac{3}{4} < 1$, $\left(\dfrac{3}{4}\right)^n$ approaches 1 as n gets large. Thus

$$\frac{3}{2}\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = \frac{3}{2}\cdot 0 = 0$$

g) $a_n = \dfrac{3^{n-1}}{2^{n+3}}$

Solution:

$$\lim_{n\to\infty} \frac{3^{n-1}}{2^{n+3}} = \lim_{n\to\infty} \frac{\frac{1}{3}\cdot 3^n}{8\cdot 2^n} = \frac{1}{24}\lim_{n\to\infty} \frac{3^n}{2^n} = \frac{1}{24}\lim_{n\to\infty} \left(\frac{3}{2}\right)^n$$

Since $\dfrac{3}{2} > 1$, $\left(\dfrac{3}{2}\right)^n$ diverges to infinity.

h) $a_n = n\pi \cos\left(\dfrac{n\pi}{2}\right)$

Solution: $\cos\left(\dfrac{n\pi}{2}\right)$ is 1 or -1, depending on whether n is even or odd. $n\pi$ diverges to infinity as n gets large. Thus $a_n = n\pi \cos\left(\dfrac{n\pi}{2}\right)$ has larger and larger absolute value but also alternating sings. A sequence like that diverges.

i) $a_n = 8^{1/n}$

Solution: $f(x) = 8^x$ is continuous on $\mathbb{R}$.

$$\lim_{n\to\infty} 8^{1/n} = 8^{\lim\limits_{n\to\infty}(1/n)} = 8^0 = 1$$

j) $a_n = \dfrac{\ln n}{\ln 2n}$

Solution: $\lim\limits_{n\to\infty} \ln n = \infty$ and so $\lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0$

$$\lim_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim_{n\to\infty} \frac{\ln n}{\ln 2 + \ln n} = \lim_{n\to\infty} \frac{1}{\frac{\ln 2}{\ln n} + 1} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} \frac{\ln 2}{\ln n} + \lim\limits_{n\to\infty} 1} = \frac{1}{0+1} = 1$$

k) $a_n = \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right)$

Solution:

$$\lim_{n\to\infty}\left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2}\right)\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2}\right)\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

l) $a_n = \frac{n!}{10^n}$

Solution: This sequence diverges to infinity. Imagine that n is very large and look at how a_{n+1} is related to a_n. The denominator is multiplied by 10 (fixed) while the numerator is multiplied by $n+1$ (indefinitely large).

$$a_{n+1} = a_n \cdot \frac{n+1}{10}$$

After $n = 20$, every term is over twice as large as the previous term. That is, in $n > 20$, then

$$a_{21} = a_{20}\cdot\frac{21}{10} > a_{20}\cdot\frac{20}{10} = 2a_{20}$$

and

$$a_{22} = a_{21}\cdot\frac{22}{10} > 2a_{21} > 4a_{20}$$

and

$$a_{20+k} > 2^k a_{21}$$

Then $a_{n+k} > 2^k a_n$ and so this sequence diverges to infinity.

m) $a_n = \frac{5^n}{n!}$

Solution: This sequence converges to zero. Imagine that n is very large and look at how a_{n+1} is related to a_n. The numerator is multiplied by 5 (fixed) while the denominator is multiplied by $n+1$ (indefinitely large). If $n > 10$, then

$$a_{11} = \frac{5}{11}a_{10} < \frac{5}{10}a_{10} = \frac{1}{2}a_{10}$$

and

$$a_{12} = \frac{5}{12}a_{11} < \frac{5}{10}a_{11} = \frac{1}{2}a_{11} = \frac{1}{4}a_{10}$$

Thus

$$a_{10+k} < \left(\frac{1}{2}\right)^k a_{10}$$

Consider now the constant zero sequence and $b_n = \left(\frac{1}{2}\right)^n a_{10}$. Both sequences converge to zero and

$$0 \le a_m \le \left(\frac{1}{2}\right)^m a_{10}$$

So a_n converges to zero by the sandwich theorem.

n) $a_n = (-1)^n \dfrac{\sin n}{n^2}$

Solution: Converges to zero by the sandwich theorem.

o) $a_n = \lim\limits_{n\to\infty} \sqrt[n]{2n}$

Solution :

$$\lim_{n\to\infty} \sqrt[n]{2n} = \lim_{n\to\infty} (2n)^{1/n} = \lim_{n\to\infty} e^{\ln(2n)^{1/n}} = \lim_{n\to\infty} e^{(1/n)\ln(2n)}$$

Looking at the exponent,

$$\lim_{n\to\infty} \frac{1}{n}\ln(2n) = \lim_{n\to\infty} \frac{\ln(2n)}{n} = \lim_{x\to\infty} \frac{\ln(2x)}{x}$$

This is an indeterminate of the form $\dfrac{\infty}{\infty}$ so we can apply l'Hôpital's rule.

$$\lim_{x\to\infty} \frac{\ln(2x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Since $f(x) = e^x$ is continuous on $\mathbb{R}$, we have that

$$\lim_{n\to\infty} e^{(1/n)\ln(2n)} = e^{\lim\limits_{n\to\infty}(1/n)\ln(2n)} = e^0 = 1$$

2. See sequences - part 3

3. Assume that $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$ and $\lim\limits_{n\to-\infty}\left(1+\dfrac{1}{n}\right)^n = e$ and compute each of the following limits.

a) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+2}$

Solution :

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2} = e\cdot 1 = e$$

Solution 2:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+2} = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^{n+2}} = \lim_{n\to\infty} e^{(n+2)\ln\left(1+\frac{1}{n}\right)} = e^{\lim\limits_{n\to\infty}(n+2)\ln\left(1+\frac{1}{n}\right)} = e^{\lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{1/(n+2)}}$$

Consider now the exponent:

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n+2}} = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x+2}}$$

This is an indeterminant of the form $\dfrac{0}{0}$ and so we can apply l'Hôpital's rule.

$$\lim_{x\to\infty}\frac{\ln\left(1+\dfrac{1}{x}\right)}{\dfrac{1}{x+2}} = \lim_{x\to\infty}\frac{\dfrac{1}{1+\dfrac{1}{x}}\cdot -\dfrac{1}{x^2}}{-\dfrac{1}{(x+2)^2}} = \lim_{x\to\infty}\frac{\dfrac{1}{x^2+x}}{\dfrac{1}{(x+2)^2}} = \lim_{x\to\infty}\frac{(x+2)^2}{x^2+x} = \lim_{x\to\infty}\frac{x^2+4x+4}{x^2+x}$$

$$= \lim_{x\to\infty}\frac{1+\dfrac{4}{x}+\dfrac{4}{x^2}}{1+\dfrac{1}{x}} = 1$$

Thus the limit is e since

$$e^{\lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{1/(n+2)}} = e^1 = e$$

b) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$

Solution :

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^2 = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right)^2 = e^2$$

Solution 2:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^{2n}} = \lim_{n\to\infty} e^{2n\ln\left(1+\frac{1}{n}\right)} = e^{\lim\limits_{n\to\infty} 2n\ln\left(1+\frac{1}{n}\right)}$$

Looking at the exponent,

$$\lim_{n\to\infty} 2n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\dfrac{1}{n}\right)}{\dfrac{1}{2n}} = \lim_{x\to\infty}\frac{\ln\left(1+\dfrac{1}{x}\right)}{\dfrac{1}{2x}}$$

This is an indeterminant of the form $\dfrac{0}{0}$ and so we can apply l'Hôpital's rule.

$$\lim_{x\to\infty}\frac{\ln\left(1+\dfrac{1}{x}\right)}{\dfrac{1}{2x}} = \lim_{x\to\infty}\frac{\dfrac{1}{1+\dfrac{1}{x}}\cdot -\dfrac{1}{x^2}}{-\dfrac{1}{2x^2}} = \lim_{x\to\infty}\frac{2}{1+\dfrac{1}{x}} = 2$$

Thus the sequence converges to e^2.

c) $\displaystyle\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{n}$

Solution :

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{5}}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{5}}\right)^{n\cdot\frac{5}{5}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}}\right]^5$$

$$= \left[\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}}\right]^5 = e^5$$

Solution 2:

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{5}{n}\right)^n} = \lim_{n\to\infty} e^{n\ln\left(1+\frac{5}{n}\right)} = e^{\lim\limits_{n\to\infty} n\ln\left(1+\frac{5}{n}\right)}$$

Looking at the exponent,

$$\lim_{n\to\infty} n\ln\left(1+\frac{5}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{5}{n}\right)}{\frac{1}{n}} = \lim_{x\to\infty}\frac{\ln\left(1+\frac{5}{x}\right)}{\frac{1}{x}}$$

This is an indeterminant of the form $\frac{0}{0}$ and so we can apply l'Hôpital's rule.

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{5}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{5}{x}}\cdot -\frac{5}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{5}{1+\frac{5}{x}} = 5$$

Thus the sequence converges to e^5.

d) $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n$

Solution :

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{-n}\right)^{n\cdot\frac{-1}{-1}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{-n}\right)^{-n}\right]^{-1}$$

Define $m=-n$

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{-n}\right)^{-n}\right]^{-1} = \left[\lim_{m\to-\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{-1} = e^{-1}$$

Solution 2:

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1-\frac{1}{n}\right)^n} = \lim_{n\to\infty} e^{n\ln\left(1-\frac{1}{n}\right)} = e^{\lim\limits_{n\to\infty} n\ln\left(1-\frac{1}{n}\right)}$$

Looking at the exponent,

$$\lim_{n\to\infty} n\ln\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}$$

This is an indeterminant of the form $\frac{0}{0}$ and so we can apply l'Hôpital's rule.

$$\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1-\frac{1}{x}}\cdot\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-1}{1-\frac{1}{x}} = -1$$

Thus the sequence converges to e^{-1}.

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