The real numbers has the **completeness property**: If a non-empty set of real numbers is bounded above, then there exists a least upper bound; if a non-empty set of real numbers is bounded below, then there exists a greatest lower bound. (This is an axiom of the real numbers, and this is the one that distinguishes the set of rational numbers from the set of real numbers.)

Definition: A **sequence** is a list of numbers $a_1, a_2, a_3, ..., a_n, ...$ in a given order. The numbers a_n are **terms** of the sequence. The integer k is called the index of the term a_k.

An infinite sequence is a function with domain $\mathbb{N}$. We may start labeling at a number greater than 1.

We can describe sequences by writing rules

$$a_n = \sqrt{n} \quad b_n = (-1)^{n+1}\frac{1}{n} \quad c_n = \frac{n-1}{n} \quad d_n = (-1)^{n+1}$$

or listing the first few terms

$$\{a_n\} = \left\{1, \sqrt{2}, \sqrt{3}, ..., \sqrt{n}, ...\right\} \qquad \{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ..., \frac{n-1}{n}, ...\right\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, ..., (-1)^{n+1}\frac{1}{n}, ...\right\} \qquad \{d_n\} = \{1, -1, 1-1, ..., (-1), ...\}$$

Definition: The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there exists an integer N such that for all n,

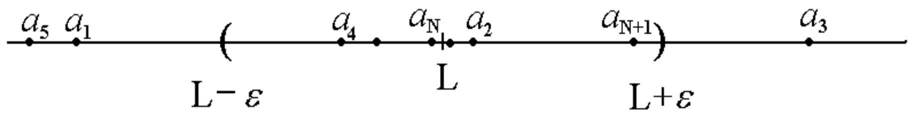
$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon.$$

If no such number L exists, we say $\{a_n\}$ **diverges**.
If $\{a_n\}$ converges to L, we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ and call L the **limit** of the sequence.

Notice that $|a_n - L| < \varepsilon$ and $L - \varepsilon < a_n < L + \varepsilon$ are equivalent statements.

$$\begin{aligned} L - \varepsilon &< a_n < L + \varepsilon \\ -\varepsilon &< a_n - L < \varepsilon \\ |a_n - L| &< \varepsilon \end{aligned}$$

This definition is fundamental to our material. We can think of a convergent sequences as one whose elements are eventually arbitrarily close to its limit. For any positive value of ε (think of ε as the error), all but finitely many elements (think of them as the first few) of the sequence are inside an $\varepsilon-$neighborhood of L.



If the sequence $\{a_n\}$ converges to L, then no matter how small ε is, all but a finitely many terms of the sequence will fall inside the ε-neightborhood of L. For every value of ε, there is generally a different value of N.

Example 1. a) $\lim_{n\to\infty} \frac{1}{n} = 0$

proof: Let $\varepsilon > 0$ be given. Define $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$.

If $n > N$, then $n > N > \frac{1}{\varepsilon}$ and so $\frac{1}{n} < \varepsilon$.

b) the constant sequnece $\{a_n\} = \{c, c, c, ...\}$. Clearly $a_n \to c$.
proof: Let $\varepsilon > 0$ be given. Then $N = 1$ will do, because for all $n > 1$ we will have that $|c - c| = 0 < \varepsilon$.

Example 2. $\{1, -1, 1-, 1, 1, -1....\}$ diverges.
proof: Suppose for a contradiction that such a number L exists. Let $\varepsilon = \frac{1}{3}$.

There exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \frac{1}{3}$. Since 1 occurs in the sequence at arbitrarily high index, it must be that

$$\begin{aligned} |1 - L| &< \frac{1}{3} \\ -\frac{1}{3} &< L - 1 < \frac{1}{3} \\ \frac{2}{3} &< L < \frac{4}{3} \end{aligned}$$

Since -1 occurs in the sequence at arbitrarily high index, it also must be that

$$\begin{aligned} |-1 - L| &< \frac{1}{3} \\ -\frac{1}{3} &< L + 1 < \frac{1}{3} \\ -\frac{4}{3} &< L < -\frac{2}{3} \end{aligned}$$

There is no number L with $\frac{2}{3} < L < \frac{4}{3}$ and $-\frac{4}{3} < L < -\frac{2}{3}$, so the sequence diverges.

Example 3. The sequence $\{\sqrt{n}\}$ diverges differently.

Definition: The sequence $\{a_n\}$ **diverges to infinity** if for every real number M there exists an integer N such that for all n,

$$\text{if } n > N \text{ then } a_n > M.$$

We denote this as $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$.

Similarly, the sequence $\{a_n\}$ **diverges to negative infinity** if for every real number m there exists an integer N such that for all n,

$$\text{if } n > N \text{ then } a_n < m.$$

We denote this as $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$.

The sequence $\{\sqrt{n}\}$ diverges to infinity. The sequence $\{1, 0, 2, 0, 3, 0, ...\}$ diverges, but does not diverge to infinity or negative infinity.

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that A and B are real numbers such that $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$. Then

1. Sum Rule: $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$
2. Difference Rule: $\lim\limits_{n\to\infty} (a_n - b_n) = A - B$
3. Constant Multiple Rule: $\lim\limits_{n\to\infty} (ca_n) = cA$ for all $c \in \mathbb{R}$
4. Product Rule: $\lim\limits_{n\to\infty} (a_n b_n) = AB$
5. Quotient Rule: $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ if $B \neq 0$

Example 4.

a) $\lim\limits_{n\to\infty} \frac{2}{n} = \lim\limits_{n\to\infty} 2 \cdot \frac{1}{n} = 2 \lim\limits_{n\to\infty} \frac{1}{n} = 2 \cdot 0 = 0$

b) $\lim\limits_{n\to\infty} \frac{n+1}{n} = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right) = \lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n} = 1 + 0 = 1$

c) $\lim\limits_{n\to\infty} \left(-\frac{3}{n^2}\right) = -3 \lim\limits_{n\to\infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right) = -3 \lim\limits_{n\to\infty} \frac{1}{n} \cdot \lim\limits_{n\to\infty} \frac{1}{n} = -3 \cdot 0 \cdot 0 = 0$

d) $\lim\limits_{n\to\infty} \frac{3 - 2n^4}{7n^4 + 2} = \lim\limits_{n\to\infty} \frac{\frac{3}{n^4} - 2}{7 + \frac{2}{n^4}} = \frac{\lim\limits_{n\to\infty} \left(\frac{3}{n^4} - 2\right)}{\lim\limits_{n\to\infty} \left(7 + \frac{2}{n^4}\right)} = \frac{-2}{7} = -\frac{2}{7}$

Theorem (The Sandwich Theorem for Sequences): Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$. Suppose that there exists N positive integer such that for all $n > N$,

$$a_n \leq b_n \leq c_n$$

then b_n converges to L.

Consequence: If $|b_n| \leq c_n$ and $c_n \to 0$, then $b_n \to 0$.

Example 5.

a) $\frac{\sin n}{n} \to 0$ since $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ and $\frac{1}{n} \to 0$ and $-\frac{1}{n} \to 0$

b) $\frac{(-1)^n}{n^2} \to 0$ since $-\frac{1}{n^2} \leq \frac{(-1)^n}{n^2} \leq \frac{1}{n^2}$

Some sequences are defined **recursively**. Recursive definitions enable us to compute the first, second, third, ... nth term, but we cannot compute the nth term without first computing the first $n - 1$.
The **Fibonacci sequence** is a perfect example for this.

Definition: The **Fibonacci sequence** is defined recursively as

$$F_1 = 1, \quad F_2 = 1, \quad \text{and for all } n \in \mathbb{N}, \quad F_{n+2} = F_n + F_{n+1}$$

The first few terms of the Fibonacci Sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$ The explicit formula for the nth term of this sequence is a very interesting formula.

Definition: A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** for the sequence $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is called a **lower bound** for the sequence $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and from below, we say that $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, we say that $\{a_n\}$ is an **unbounded** sequence.

Example 6. a) The sequence $1, 4, 9, 16, ...$ is bounded from below but not from above. 1 is the greatest lower bound for the sequence.

b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ..., \frac{n}{n+1}$ is bounded. $\frac{1}{2}$ is the greatest lower bound and 1 is the lowest upper bound for this sequence.

Theorem: If $\{a_n\}$ is convergent, then a_n is bounded.

proof: Suppose that $\{a_n\}$ is a convergent sequence and $a_n \to L$. Let $\varepsilon = 1$. There exists a natural number N such that for all $n > N$,

$$L - 1 < a_n < L + 1$$

Consider now the set $\{a_1, a_2, a_3, ..., a_N\}$. Since this is a finite set, it has a lowest and greatest element. Denote these by m and M, respectively. We claim that $\min(m, L-1)$ is a lower bound for the sequence $\{a_n\}$ and $\max(M, L+1)$ is an upper bound for the sequence.

Let a_k be any term of the sequence. If $k > N$, then $L - 1 < a_k < L + 1$ and so

$$\min(m, L-1) \leq L - 1 < a_k < L + 1 \leq \max(M, L+1)$$

and if $k \leq N$, then a_k is in the set $\{a_1, a_2, a_3, ..., a_N\}$ and so

$$\min(m, L-1) \leq m < a_k < M \leq \max(M, L+1)$$

Defintion: A sequence $\{a_n\}$ is **nondecreasing** if $a_n \le a_{n+1}$ for all n. That is, $a_1 \le a_2 \le a_3 \le ...$ The sequence is **nonincreasing** if $a_n \ge a_{n+1}$ for all n. That is, $a_1 \ge a_2 \ge a_3 \ge ...$ The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

Example 7. a) The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ is nonincreasing.

b) the constant sequence $2, 2, 2, ...$ is both nonincreasing and nondecreasing.

c) the sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, ...$ is not monotonic.

Theorem: If a sequence $\{a_n\}$ is bounded from above and non-decreasing, then it is also convergent. (Similarly, if a sequence is bounded from below and non-increasing, then it is convergent.)

proof: Suppose that $\{a_n\}$ is bounded and nondecreasing. Let L be the least upper bound for the sequence. Since L is an upper bound, $a_n \le L$ for all n.

Let $\varepsilon > 0$ be given. Since L is the lowest upper bound, $L - \varepsilon$ is NOT an upper bound. This means that there exists m natural number such that $a_m > L - \varepsilon$. Since a_n is nondecreasing, all subsequent terms will have this property, i.e. for all $n > m$, $a_n \ge a_m > L - \varepsilon$. thus we have that for all $n > m$

$$L - \varepsilon < a_n \le L < L + \varepsilon$$

and so

$$L - \varepsilon < a_n < L + \varepsilon$$

and so a_n converges to L. The proof for nonincreasing sequences is similar.

Proof of the sum rule: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that A and B are real numbers such that $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$. Then $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$.

proof: Let $\varepsilon > 0$ be given. There exist N_a and N_b natural numbers such that for all $n > N_a$,

$$A - \frac{\varepsilon}{2} < a_n < A + \frac{\varepsilon}{2}$$

and for all $k > N_b$,

$$B - \frac{\varepsilon}{2} < b_k < B + \frac{\varepsilon}{2}$$

Let $N = \max(N_a, N_b)$. If $n > N$, then

$$A - \frac{\varepsilon}{2} < a_n < A + \frac{\varepsilon}{2} \text{ and } B - \frac{\varepsilon}{2} < b_n < B + \frac{\varepsilon}{2}$$

Adding these two inequalities we obtain

$$\begin{aligned} A - \frac{\varepsilon}{2} + B - \frac{\varepsilon}{2} &< a_n + b_n < A + \frac{\varepsilon}{2} + B + \frac{\varepsilon}{2} \\ A + B - \varepsilon &< a_n + b_n < A + B + \varepsilon \end{aligned}$$

Thus $a_n + b_n$ converges to $A + B$.

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