Definition: Given a sequence of numbers $\{a_n\}$, the expression

$$a_1 + a_2 + a_3 + ... + a_n + ...$$

is called an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

is called the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series converges and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Definition: **Geometric series** are of the form

$$a + ar + ar^2 + ... + ar^n + ... = \sum_{n=0}^{\infty} ar^n$$

where a and r are fixed real numbers and $a \neq 0$.

Theorem: If $r = 1$, then $s_n = na$ and if $r \neq 1$, then $s_n = a\dfrac{1-r^n}{1-r}$.

proof. Case 1. If $r = 1$, then $a_n = a + ar + ar^2 + ... + ar^{n-1} = a + a + ... + a = na$.

Case 2. If $r \neq 1$, then

$$\begin{aligned} a + ar + ar^2 + ... + ar^{n-1} &= s_n \quad \text{multiply both sides by } r \\ ar + ar^2 + ar^3 + ... + ar^n &= rs_n \quad \text{subtract} \\ &\Downarrow \\ a - ar^n &= s_n - rs_n \\ a\left(1-r^n\right) &= s_n\left(1-r\right) \qquad \text{divide by } 1-r \\ a\frac{1-r^n}{1-r} &= s_n \end{aligned}$$

Theorem: If $|r| < 1$, the geometric series $a + ar + ar^2 + ... + ar^{n-1} + ...$ converges to $\dfrac{a}{1-r}$ and if $|r| \geq 1$, the series diverges.

proof: If $r = 1$, then the sequence is constant. In this case, if $a = 0$, the sequence is the constant zero sequence. Then the series converges to zero. If $a \neq 0$, the series diverges to infinity or negative infinity, depending on the sign of a.
Suppose now that $r \neq 1$. The infinte sum is defined as the limit of the partial sums:

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r} = a\frac{1-\lim\limits_{n\to\infty} r^n}{1-r}$$

Now if $|r| > 1$, then r^n diverges and so there is no infinite sum defined. If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n = 0$ and so

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r} = a\frac{1-\lim\limits_{n\to\infty} r^n}{1-r} = a\frac{1}{1-r} = \frac{a}{1-r}$$

Sample Problems

In each case, compute the sum of the infinite series given.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{2}{3^n}$
2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+1}}{3^n}$
4. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{5^n}{3^{2n-1}}$

Practice Problems

In each case, determine whether the given geometric series converges or diverges. If converges, find its sum.

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$
2. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1}}{3^{n-1}}$
3. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{n+1}}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+1}}{5^{n-2}}$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}(-3)^{n+1}}{5^{n-2}}$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{9^n}{10^{n+1}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{(-10)^n}$
9. $\displaystyle\sum_{n=2}^{\infty} \frac{2^n 3^{n+1}}{7^{n-1}}$
10. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$

11*. $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{3^{n+1}} = \frac{1}{9} + \frac{3}{27} + \frac{5}{81} + \frac{7}{243} + ...$

Answers - Sample Problems

1.) 3   2.) 2   3.) diverges   4.) $\frac{27}{14}$

Answers - Practice Problems

1.) $\frac{1}{9}$ 2.) $\frac{18}{5}$ 3.) diverges   4.) 250   5.) diverges   6.) 1   7.) $\frac{1}{3}$ 8.) $-\frac{3}{11}$ 9.) 108

10.) $\frac{e}{e-1}$ 11.) $\frac{1}{3}$

Sample Problems - Solutions

In each case, compute the sum of the infinite series given.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{2}{3^n}$

   This sequence is $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\ldots$ Thus $a=2$ and $r=\frac{1}{3}$. The sum of the series exists and can be computed as
   $$s=\frac{a}{1-r}=\frac{2}{1-\frac{1}{3}}=\frac{2}{\frac{2}{3}}=3$$

2. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+1}$

   This sequence is $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots\ldots$ Thus $a=\frac{2}{3}$ and $r=\frac{2}{3}$. The sum of the series exists and can be computed as
   $$s=\frac{a}{1-r}=\frac{\frac{2}{3}}{1-\frac{2}{3}}=\frac{\frac{2}{3}}{\frac{1}{3}}=2$$

3. $\displaystyle\sum_{n=0}^{\infty}\frac{2^{2n+1}}{3^n}$

   Sometimes a bit of algebra helps more than writing out the first few terms immediately. We can re-write 2^{2n+1} as
   $$2^{2n+1}=2^{2n}\cdot 2=\left(2^2\right)^n\cdot 2=4^n\cdot 2$$
   and so
   $$\sum_{n=0}^{\infty}\frac{2^{2n+1}}{3^n}=\sum_{n=0}^{\infty}\frac{4^n\cdot 2}{3^n}=\sum_{n=0}^{\infty}2\left(\frac{4}{3}\right)^n$$
   This sequence is $2, \frac{8}{3}, \frac{32}{27}, \ldots\ldots$ Thus $a=2$ and $r=\frac{4}{3}$. Since $\frac{4}{3}>1$, this series diverges.

4. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{3^{2n-1}}$

$$\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{3^{2n-1}}=\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{\frac{3^{2n}}{3}}=\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 5^n}{(3^2)^n}=\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 5^n}{9^n}=\sum_{n=0}^{\infty}3\cdot\left(-\frac{5}{9}\right)^n$$

Thus $a=3$ and $r=-\dfrac{5}{9}$. Then the sum of the series exists and can be computed as

$$s=\frac{a}{1-r}=\frac{3}{1-\left(-\frac{5}{9}\right)}=\frac{3}{\frac{14}{9}}=\frac{27}{14}$$

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