Definition: **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ...$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + ... + c_n (x-a)^n + ...$$

in which the center a and the coefficients $c_0, c_1, c_2, ...$ are constants.

Example 1: The geometric series with first term 1 and common ratio x:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + ...$$

When $-1 < x < 1$, then this series converges to $\dfrac{1}{1-x}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... \qquad -1 < x < 1$$

Example 2: The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + ... + \left(-\frac{1}{2}\right)^n (x-2)^n + ...$$

is a geometric series with first term 1 and common ratio $-\dfrac{1}{2}(x-2)$. This series converges if $-1 < -\dfrac{1}{2}(x-2) < 1$. We solve this for x

$$\begin{aligned} -1 &< -\frac{1}{2}(x-2) < 1 \\ 2 &> x - 2 > -2 \\ 4 &> x > 0 \end{aligned}$$

The sum is then $\dfrac{1}{1-r} = \dfrac{1}{1 - \left(-\dfrac{1}{2}(x-2)\right)} = \dfrac{1}{1 + \dfrac{1}{2}x - 1} = \dfrac{2}{x}$ and so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \frac{(x-2)^3}{8} + ... + \left(-\frac{1}{2}\right)^n (x-2)^n \qquad \text{when } 0 < x < 4$$

So, as long as x is in the open interval $(0, 4)$, the power series becomes a convergent geometric series with sum $\dfrac{2}{x}$.

Example 3: For what values of x do the following power series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ converge?

Solution: We apply the ratio test.

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(-1)^{n+2}\dfrac{x^{n+1}}{n+1}\right|}{\left|(-1)^{n+1}\dfrac{x^n}{n}\right|} = \lim_{n\to\infty} \frac{\dfrac{|x \cdot x^n|}{n+1}}{\dfrac{|x^n|}{n}} = \lim_{n\to\infty} \left(\frac{|x| \cdot |x^n|}{n+1} \frac{n}{|x^n|}\right) = \lim_{n\to\infty} \left(|x| \cdot \frac{n}{n+1}\right) \\ &= |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| \end{aligned}$$

If $\rho < 1$, the series converges absolutely. If $\rho > 1$, the series diverges.

$$\rho < 1 \iff |x| < 1 \iff -1 < x < 1 \quad \text{and} \quad \rho > 1 \iff |x| > 1 \iff x > 1 \text{ or } x < -1$$

So the series diverges for x with $x > 1$ or $x < -1$ and converges absolutely for $-1 < x < 1$. The ratio test is not conclusive when $\rho = 1$, so we have to check those cases separately.

$$\rho = 1 \iff |x| = 1 \iff x = \pm 1$$

If $x = 1$, then $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^2}{3} - \frac{x^2}{4} + ...$ becomes

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1^n}{n} = 1 - \frac{1^2}{2} + \frac{1^2}{3} - \frac{1^4}{4} + ... = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

This is the alternating harmonic series, which converges conditionally. So the power series converges for $x = 1$.

If $x = -1$, then $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^2}{3} - \frac{x^2}{4} + ...$ becomes

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n} = (-1) - \frac{(-1)^2}{2} + \frac{(-1)^2}{3} - \frac{(-1)^4}{4} + ... = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - .....$$

This is the harmonic series, multiplied by -1. Since the harmonic series diverges, so does its oppsite. Thus the power series diverges for $x = -1$. So the power series converges on the interval $(-1, 1]$.

Theorem: (The Convergence Theorem for Power Series) If a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Proof. Part 1. Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$.

Since $\displaystyle\sum_{n=0}^{\infty} a_n c^n$ converges, the nth term approaches zero. Therefore, there exists $N \in \mathbb{N}$ so that for all $n > N$, $|a_n c^n| < 1$. Then

$$\begin{aligned} |a_n c^n| &< 1 \\ |a_n|\,|c|^n &< 1 \\ |a_n| &< \frac{1}{|c|^n} \end{aligned}$$

Now let x be given with $|x| < |c|$. Then $\left|\dfrac{x}{c}\right| < 1$. Let us multiply both sides of the inequality by $|x|^n$.

$$\begin{aligned} |a_n|\,|x^n| &< \frac{|x|^n}{|c|^n} \\ |a_n x^n| &< \left|\frac{x}{c}\right|^n \end{aligned}$$

$\displaystyle\sum_{n=0}^{\infty}\left|\frac{x}{c}\right|^n$ converges because it is a geometric series with $|r|<1$. By the comparison test, $\displaystyle\sum_{n=0}^{\infty}|a_n x^n|$ converges and so $\displaystyle\sum_{n=0}^{\infty}a_n x^n$ converges absolutely.

Part 2. Suppose $\displaystyle\sum_{n=0}^{\infty}a_n x^n$ diverges at some $x=d$ and let x be given with $|x|>|d|$.

If $\displaystyle\sum_{n=0}^{\infty}a_n x^n$ converges at x with $|x|>|d|$, by Part 1, this implies that the series converges absolutely at $x=d$. This is impossible, so the power series does not converge at x. This completes our proof. ■

Corollary: The convergence of series $\displaystyle\sum_{n=0}^{\infty}c_n(x-a)^n$ is described by one of the following three cases:
1.) The series converges on a and diverges at every $x\neq a$.
2.) There exists a positive number R so that the series converges absolutely on the interval $(a-R,a+R)$ and diverges on $(-\infty,a-R)\cup(a+R,\infty)$. At the endpoints, $x=a-R$ and $x=a+R$, the series may converge or diverge.
3.) The series converges absolutely on $\mathbb{R}$.

Theorem: If a power series $\displaystyle\sum_{n=0}^{\infty}a_n x^n$ converges absolutely for $|x|<R$, then $\displaystyle\sum_{n=0}^{\infty}a_n(f(x))^n$ converges absolutely for any continuous function f on $|f(x)|<R$.

Theorem: (Term-by term Differentiation Theorem) If a power series $\displaystyle\sum_{n=0}^{\infty}c_n(x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n \text{ on the interval } a-R<x<a+R$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the series term by term: $f'(x)=\displaystyle\sum_{n=0}^{\infty}c_n n(x-a)^{n-1}$ and $f''(x)=\displaystyle\sum_{n=0}^{\infty}c_n n(n-1)(x-a)^{n-2}$ and so on. Each of these derived series converges at every point inside the interval $a-R<x<a+R$.

Theorem: (Term-by term Integration Theorem) Suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a - R < x < a + R$ where $R > 0$. Then $\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Please note that the inside of the interval behaves different from the endpoints. When differentiating, integrating, or multiplying a power series to obtain another one, the resulting series converge inside of the interval $(a - R, a + R)$. But there is no general rule how the endpoint will behave. We always need to check the endpoints of the interval of convergence.

For example, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + ...$ converges on the open interval $(-1, 1)$ but not at either of its endpoints, $x = 1$ and $x = -1$. After integrating both sides, we obtain the power series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}........$ Based on the theorems listed above, this series converges on $(-1, 1)$. However, we should check the enpoints of this interval. If $x = -1$, then this series becomes

$$(-1) - \frac{(-1)^2}{2} + \frac{(-1)^3}{3} - \frac{(-1)^4}{4}....... = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - .... \text{ which diverges}$$

But if $x = 1$, then the series becomes

$$1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4}....... = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... \text{ which converges conditionally}$$

and so the interval of convergence is actually $(-1, 1]$.

List of important power series. Make sure you can quickly derive or memorize the following power series: The geometric series gave us

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... \quad \text{on } (-1, 1)$$

When we use common ratio of $-x$ instead of x, we obtain another another geometric series,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... \quad \text{on } (-1, 1)$$

When we integrate both sides, we obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}....... \quad \text{on } (-1, 1]$$

When we start with a geometric series with common ratio $-x^2$, we obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^3 - x^4 + ... \quad \text{on } (-1, 1)$$

and when we integrate both sides, we obtain

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \quad \text{on } (-1, 1]$$

Sample Problems

1. Find all values of x for which the series converges. For what values will the series converge conditionally?

   a) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$ b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{x^{n}}{n!}$ d) $\displaystyle\sum_{n=1}^{\infty}\frac{1+2+3+...+n}{1^2+2^2+3^2+...+n^2}x^n$

2. Find a power series that converges to $\ln(1-x)$.

3. a) Find a series that converges to $\dfrac{1}{1+x^2}$.

   b) Find a series that converges to $\tan^{-1}x$.

4. Find the power series that converges to $\dfrac{1}{1+x}$ and use that to find the power series that converges to $\ln(1+x)$.

5. Consider $\displaystyle\sum_{n=0}^{\infty}\frac{n^2}{2^n}$

   a) Start with $\dfrac{1}{1-x}=1+x+x^2+x^3+.....$ on $(-1,1)$ and differentiate both sides and multiply by x

   b) differentiate both sides and multiply by x again

   c) Set $x=\dfrac{1}{2}$

6. Find the radius of convergence. Then, within the radius of convergence, find the function to which the power series converges.

   a) $\displaystyle\sum_{n=0}^{\infty}3^n x^n$ b) $\displaystyle\sum_{n=0}^{\infty}(\ln x)^n$

7. Use power series to find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n}$

Solutions - Sample Problems

1. Find all values of x for which the series converges. For what values will the series converge conditionally?

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$ $(-1,1)$

Apply the ratio test:

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{2n+2}}{n+1}}{\frac{x^{2n}}{n}}\right|=\lim_{n\to\infty}\left|\frac{x^2\cdot x^{2n}}{n+1}\frac{n}{x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{x^2 n}{n+1}\right|=x^2\cdot\lim_{n\to\infty}\frac{n}{n+1}=|x|$$

So the series converges absolutely on $(-1,1)$. We check the endpoints: at $x=-1$, $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$ becomes

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(-1)^{2n}}{n}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left((-1)^2\right)^n}{n}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}=\sum_{n=0}^{\infty}-\frac{1}{n}=-\sum_{n=0}^{\infty}\frac{1}{n}$$

this is the harmonic series which diverges. At $x=1$, the series $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$ becomes

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1^{2n}}{n}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}=\sum_{n=0}^{\infty}-\frac{1}{n}=-\sum_{n=0}^{\infty}\frac{1}{n}$$

so the series diverges at $x=1$ as well. So the radius of convergence is $(-1,1)$.

b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ $[-1,1]$

use ratio test:

$$\begin{aligned}\rho&=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)-1}}{2(n+1)-1}}{\frac{x^{2n-1}}{2n-1}}\right|=\lim_{n\to\infty}\left|\frac{x^{2(n+1)-1}}{2(n+1)-1}\frac{2n-1}{x^{2n-1}}\right|\\&=\lim_{n\to\infty}\left|\frac{x^{2n+1}}{2n+1}\frac{2n-1}{x^{2n-1}}\right|=\lim_{n\to\infty}\left|\frac{x^2\cdot x^{2n-1}}{2n+1}\frac{2n-1}{x^{2n-1}}\right|=x^2\lim_{n\to\infty}\frac{2n-1}{2n+1}=x^2\end{aligned}$$

So the series converges absolutely on $(-1,1)$. We check the endpoints: at $x=\pm1$, the series converges conditionally.

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{x^{n}}{n!}$ $\mathbb{R}$

use ratio test:

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{x\cdot x^n}{(n+1)\,n!}\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0$$

So the series converges absolutely on $\mathbb{R}$.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2+3+...+n}{1^2+2^2+3^2+...+n^2} x^n \qquad [-1,1)$

$$a_n = \frac{1+2+3+...+n}{1^2+2^2+3^2+...+n^2} x^n = \frac{\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}} x^n = \frac{n(n+1)}{2} \cdot \frac{6}{n(n+1)(2n+1)} x^n = \frac{3}{2n+1}\, x^n$$

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3}{2(n+1)+1}\, x^{n+1}}{\frac{3}{2n+1}\, x^n}\right| = \lim_{n\to\infty}\left|\frac{3\, x \cdot x^n}{2n+3} \cdot \frac{2n+1}{3\, x^n}\right| \\ &= \lim_{n\to\infty}\left|x\, \frac{2n+1}{2n+3}\right| = |x| \lim_{n\to\infty}\frac{2n+1}{2n+3} = |x|\end{aligned}$$

So the series converges absolutely on $(-1,1)$. We check at the endpoints: at $x=1$, we get a series that is divergent. We prove this by the comparison test:

$\displaystyle\sum_{n=1}^{\infty}\frac{3}{2n+1} \geq \sum_{n=1}^{\infty}\frac{3}{2n+2} = \frac{3}{2}\sum_{n=2}^{\infty}\frac{1}{n}$ and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ is divergent. At the other endpoint, $x=-1$, the series is conditionally convergent by Leibniz's test.

$$\sum_{n=1}^{\infty}\frac{3}{2n+1}(-1)^n = -1 + \frac{3}{5} - \frac{3}{7} + \frac{3}{9} - ......$$

2. Find a power series that converges to $\ln(1-x)$.

Start with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... \quad \text{for all } x \in (-1,1)$$

integrate both sides: $-\ln(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + ...$ on $(-1,1)$

and so $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - ......$ on $(-1,1)$

3. a) Find a series that converges to $\dfrac{1}{1+x^2}$.

Solution: in $\dfrac{a}{1-r}$, we make $r=-x^2$, so $s = \dfrac{1}{1-(-x^2)} = \dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - ...... \quad \text{on } (-1,1)$$

b) Find a series that converges to $\tan^{-1} x$.

integrate both sides:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \quad \text{on } (-1,1)$$

4. Find the power series that converges to $\dfrac{1}{1+x}$ and use that to find the power series that converges to $\ln(1+x)$.

$$\begin{aligned}\frac{1}{1+x} &= 1 - x + x^2 - x^3 + ... \quad \text{on } (-1,1) \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}....... \quad \text{on } (-1,1)\end{aligned}$$

5. Consider $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

a) Start with $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots..$ on $(-1,1)$ and differentiate both sides and multiply by x

$$\frac{1}{(x-1)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots..$$
$$\frac{x}{(x-1)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots..$$

b) differentiate both sides and multiply by x again

$$\frac{-x-1}{(x-1)^3} = 1 + 4x + 9x^2 + 16x^3 + \ldots...$$
$$\frac{x(-x-1)}{(x-1)^3} = x + 4x^2 + 9x^3 + 16x^4 + \ldots...$$

c) Set $x = \dfrac{1}{2}$

$$x + 4x^2 + 9x^3 + 16x^4 + \ldots... = \frac{x(-x-1)}{(x-1)^3}$$
$$\frac{1}{2} + 4\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right)^3 + 16\left(\frac{1}{2}\right)^4 + \ldots... = \sum_{n=0}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}\left(-\frac{1}{2}-1\right)}{\left(\frac{1}{2}-1\right)^3} = 6$$

6. Find the radius of convergence. Then, within the radius of convergence, find the function to which the power series converges.

a) $\displaystyle\sum_{n=0}^{\infty} 3^n x^n \qquad f(x) = \frac{1}{1-3x}$ on $-\dfrac{1}{3} < x < \dfrac{1}{3}$

This is clearly a geometric series with $r = 3x$. So it converges when $|r| < 1$, that is when $-\dfrac{1}{3} < x < \dfrac{1}{3}$. Then the sum is $\dfrac{1}{1-3x}$.

b) $\displaystyle\sum_{n=0}^{\infty} (\ln x)^n$

This is clearly a geometric series with $r = \ln x$. So it converges when $|r| < 1$, that is when $-1 < \ln x < 1$ which is when $\dfrac{1}{e} < x < e$. Then the sum is $\dfrac{1}{1-\ln x}$.

7. Use power series to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n}$.

Solution: Start with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots. \quad \text{on } (-1,1)$$

and integrate both sides:

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots. \quad \text{on } (-1,1)$$

Now evaluate at $x = \frac{1}{2}$

$$
\begin{aligned}
-\ln\left(1-\frac{1}{2}\right) &= \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^4}{4} + ... \quad \text{on } (-1,1) \\
-\ln\left(\frac{1}{2}\right) &= \ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + ... + \frac{\left(\frac{1}{2}\right)^n}{n} + ...
\end{aligned}
$$

Also note that $-\ln\left(\frac{1}{2}\right) = \ln\left(\left(\frac{1}{2}\right)^{-1}\right) = \ln 2$. So $\displaystyle\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n} = \ln 2$.

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