Definition: Given a sequence of numbers $\{a_n\}$, the expression

$$a_1 + a_2 + a_3 + ... + a_n + ...$$

is called an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

is called the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series converges and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example 1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

Proof: The series is $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + ...$ Consider the sequence $\{s_n\}$ of partial sums.

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\ s_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} \end{aligned}$$

$$\begin{aligned} s_4 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5} \\ s_5 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6} \\ s_6 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{6}{7} \end{aligned}$$

It appears that $s_n = \dfrac{n}{n+1}$. If that was so, then the sum of the series can be easily found as the limit of the sequence of partial sums:

$$s = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

This is the case indeed. We can prove it in general using partial fractions:

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

and so

$$\begin{aligned}
s_1 &= a_1 = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2} \\
s_2 &= a_1 + a_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\
s_3 &= a_1 + a_2 + a_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4} \\
s_n &= a_1 + a_2 + \ldots + a_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}
\end{aligned}$$

In this case, each partial sum of the series has only a fixed number of terms after cancellation. We call such a series a **telescoping sum**.

Definition: **Geometric series** are of the form

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots . = \sum_{n=1}^{\infty} ar^{n-1}$$

where a and r are fixed real numbers and $a \neq 0$.

Theorem: If $|r| < 1$, the geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ converges to $\dfrac{a}{1-r}$ and if $|r| \geq 1$, the series diverges.

Example 2) Determine whether the given series converges or diverges. If it converges, find the sum of the series.

a) $\displaystyle\sum_{n=0}^{\infty} 2^{n-1} \cdot 3^{2-n}$

We start by re-writing a_n

$$a_n = 2^{n-1} \cdot 3^{2-n} = \frac{2^n}{2} \cdot \frac{9}{3^n} = \frac{9}{2}\left(\frac{2}{3}\right)^n$$

We can now determine that $a = \dfrac{9}{2}$ and $r = \dfrac{2}{3}$. Since $-1 < r < 1$, the series converges and its sum is

$$s = \frac{a}{1-r} = \frac{\frac{9}{2}}{1 - \frac{2}{3}} = \frac{27}{2}$$

b) $\displaystyle\sum_{n=3}^{\infty} 3^{2n-1} \cdot (-5)^{2-n}$

We start by re-writing a_n

$$a_n = 3^{2n-1} \cdot (-5)^{2-n} = \frac{9^n}{3} \cdot \frac{25}{(-5)^n} = \frac{25}{3}\left(-\frac{9}{5}\right)^n$$

We can now determine that $a = \dfrac{25}{3}$ and $r = -\dfrac{9}{5}$. Since $-r < -1$, the series diverges and so the sum is not defined. There are more examples in the separate handout dedicated to geometric series.

Theorem: If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Proof: Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a convergent series with sum $s \in \mathbb{R}$. Then $\displaystyle\lim_{n\to\infty} s_n = s$. Clearly $a_n = s_n - s_{n-1}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Theorem: (nth term test) If $\displaystyle\lim_{n\to\infty} a_n$ doesn't exist or exists but is not zero, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

The "smallness" of the nth term is a necessary but not sufficient condition of the convergence of the series. For example, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though the nth term, a_n approaches zero. The logical conclusion is that the nth term test can only be used to prove divergence of a series.

Example 3) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{2n+1}$ diverges.

Proof: $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-2}{2n+1} = \lim_{n\to\infty} \frac{1-\frac{2}{n}}{2+\frac{1}{n}} = \frac{1}{2}$. Since the nth term does not approach zero, the series diverges.

Theorem: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1) Sum Rule $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2) Difference Rule $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3) Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA$ for any $k \in \mathbb{R}$

These properties can be proved using the properties of limits of sequences. Consequences:

1) Every non-zero constant multiple of a divergent series diverges.
2) If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

Example 4.) Compute the sum of each of the following series or state if the series diverges.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{7 \cdot 2^n - 3^n}{5^n}$

Solution:

$$\sum_{n=0}^{\infty}\frac{7\cdot 2^n-3^n}{5^n}=\sum_{n=0}^{\infty}\left(\frac{7\cdot 2^n}{5^n}-\frac{3^n}{5^n}\right)=\sum_{n=0}^{\infty}7\left(\frac{2^n}{5^n}\right)-\sum_{n=0}^{\infty}\left(\frac{3^n}{5^n}\right)=\sum_{n=0}^{\infty}7\left(\frac{2}{5}\right)^n-\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$$

We separately compute the sums of the geometric series. In the first series, $a=7$ and $r=\dfrac{2}{5}$ and in the second, $a=1$ and $r=\dfrac{3}{5}$.

$$s_1=\frac{7}{1-\frac{2}{5}}=\frac{35}{3}\quad\text{and}\quad s_2=\frac{1}{1-\frac{3}{5}}=\frac{5}{2}$$

and so the difference of the two series is $\dfrac{35}{3}-\dfrac{5}{2}=\dfrac{55}{6}$.

b) $\displaystyle\sum_{n=0}^{\infty}e^n+e^{-n}$

Solution:

$$\sum_{n=0}^{\infty}e^n+e^{-n}=\sum_{n=0}^{\infty}e^n+\sum_{n=0}^{\infty}e^{-n}$$

The first series diverges because it is a geometric series with $r=e>1$. Thus the sum of the two series diverges as well.

Recall the Monotonic Sequence Theorem: If a non-decreasing sequence is bounded from above, then it is convergent. A consequence of this is the following theorem.

Theorem: A series $\displaystyle\sum_{n=1}^{\infty}a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Proof: If the sequence $\{a_n\}$ has only non-negative terms, then the sequence $\{s_n\}$ of partial sums is non-decreasing.

Example 5.) The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Proof: We group the terms into clusters that each add up to a number greater than $\dfrac{1}{2}$. Since there are infinitely many such clusters, the sequence of partial sums is not bounded from above. Thus the series diverges.

$$1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{2}{4}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\frac{4}{8}}+\underbrace{\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)}_{>\frac{8}{16}}+\ldots.$$

This is an excellent example to demonstrate that the nth term may approach zero and yet the series diverges.

Theorem: (The Integral Test) Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Proof. Suppose that the conditions hold. Suppose that $N = 1$. If the function f is decreasing, then all left-hand Riemann sums overestimate the area under the graph and all right-hand sums underestimate the same area. Consider now the following:

$a_1 + a_2 + a_3 + ... + a_n$ is a left sum for f on the interval $[1, n+1]$ with the partition $\{1, 2, 3, ..., n+1\}$

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + a_3 + ... + a_n$$

$a_2 + a_3 + ... + a_n$ is a right sum for f on $[1, n]$ with the partition $\{1, 2, 3, ..., n\}$. Thus

$$a_2 + a_3 + ... + a_n \leq \int_1^{n} f(x)\,dx$$

Add a_1 to both sides:

$$a_1 + a_2 + a_3 + ... + a_n \leq a_1 + \int_1^{n} f(x)\,dx$$

And so we have that

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + a_3 + ... + a_n \leq a_1 + \int_1^{n} f(x)\,dx$$

These inequalities are true for all n, and continue to hold as n approaches infinity.

If $\int_1^{\infty} f(x)\,dx$ is finite, then $a_1 + \int_1^{\infty} f(x)\,dx$ is an upper limit of the sequence of partial sums, and so $\sum_{n=1}^{\infty} a_n$ is finite.
If $\int_1^{\infty} f(x)\,dx$ is infinite, then $\int_1^{n+1} f(x)\,dx$ is not bounded from above and so the sequence of partial sums is also not bounded from above, thus $\sum_{n=1}^{\infty} a_n$ is infinite. Thus the series and the integral are both finite or both infinite.

Example 6.) Determine whether the given series is convergent or not.

a) $\sum_{n=1}^{\infty} \frac{1}{n}$

This will be the second time we prove that this series diverges, but it is a very famous series. It is called the harmonic series. We will use the integral test. The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ is decreasing and all terms are positive.

Also, the function $f(x) = \frac{1}{x}$ is continuous, positive, and decreasing on $[1, \infty)$, so we may apply the integral test. Since $\int_1^\infty \frac{1}{x} dx$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{n}$.

b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

We will use the integral test. The sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...$ is decreasing and all terms are positive. Also, the function $f(x) = \frac{1}{x^2}$ is continuous, positive, and decreasing on $[1, \infty)$, so we may apply the integral test. Since $\int_1^\infty \frac{1}{x^2} dx$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

c) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

We will use the integral test. The sequence $\frac{1}{3 \ln 3}, \frac{1}{4 \ln 4}, \frac{1}{5 \ln 5}, ...$ is decreasing and all terms are positive. Also, the function $f(x) = \frac{1}{x \ln x}$ is continuous, and positive on $[3, \infty)$. We may apply the integral test, but first we must prove that $f(x) = \frac{1}{x \ln x}$ is decreasing on $[3, \infty)$. We differentiate $f(x)$

$$\frac{d}{dx}\left(\frac{1}{x \ln x}\right) = \frac{d}{dx}\left((x \ln x)^{-1}\right) = -1 (x \ln x)^{-2} (\ln x + 1) = -\frac{1 + \ln x}{(x \ln x)^2}$$

Since the derivative of f is negative on $[3, \infty)$, the function is decreasing and so the conditions for applying the integral test all hold. We compute the improper integral $\int_3^\infty \frac{1}{x \ln x} dx$. We use integration by substitution to compute the indefinite integral: let $u = \ln x$ then $du = \frac{1}{x} dx$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x}\left(\frac{1}{x} dx\right) = \int \frac{1}{u} du = \ln |u| + C = \ln |\ln x| + C$$

Now for the improper integral:

$$\int_3^\infty \frac{1}{x \ln x} dx = \lim_{N \to \infty} \int_3^N \frac{1}{x \ln x} dx = \lim_{N \to \infty} \ln |\ln x| \Big|_3^N = \lim_{N \to \infty} (\ln \ln N - \ln \ln 3) = \infty$$

Since the integral diverges, so does the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$.

Practice Problems

Determine which of the following series converge and which diverge.

Please note that most of these problems can be solved using several different methods.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2\left(n+1\right)^2}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n+3^n}{4^n}$
5. $\displaystyle\sum_{n=0}^{\infty} \sqrt{n}$
6. $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n^2}$
8. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{e^n}$
11. $\displaystyle\sum_{n=1}^{\infty} \operatorname{sech} n$
12. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$
14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Answers - Practice Problems

1.) converges to $\frac{3}{4}$ 2.) converges to 1 3.) converges 4.) diverges 5.) diverges 6.) diverges

7.) converges 8.) diverges 9.) converges 10.) converges 11.) converges 12.) diverges

13.) converges 14.) diverges

Solutions

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$ converges to $\frac{3}{4}$, it is a telescoping sum

$$a_n = \frac{1}{n^2+2n} = \frac{1}{n(n+2)} = \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\begin{aligned}
s_1 &= a_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) = \frac{1}{3} \\
s_2 &= a_1 + a_2 = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)\right] = \frac{11}{24} \\
s_3 &= a_1 + a_2 + a_3 = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)\right] = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}\right) = \frac{21}{40} \\
s_3 &= a_1 + a_2 + a_3 = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)\right] = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right) = \frac{17}{30} \\
s_n &= a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \\
&= \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots.. + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right] \\
&= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)
\end{aligned}$$

Thus $s_n = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$ and so

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{4}$$

2. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ converges to 1

Solution: We use partial fractions to decompose $a_n = \frac{2n+1}{n^2(n+1)^2}$

$$\begin{aligned}
\frac{2n+1}{n^2(n+1)^2} &= \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2} \\
\frac{2n+1}{n^2(n+1)^2} &= \frac{An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2}{n^2(n+1)^2} \\
2n+1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2
\end{aligned}$$

Let $n=0$

$$\begin{aligned} 2n+1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ 1 &= B \end{aligned}$$

Let $n=-1$

$$\begin{aligned} 2n+1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ -1 &= D \end{aligned}$$

Let $n=1$

$$\begin{aligned} 2n+1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ 3 &= 4A+4+2C-1 \\ 0 &= 4A-2C \\ C &= 2A \end{aligned}$$

Let $n=2$

$$\begin{aligned} 2n+1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ 5 &= 18A+9+12C-4 \\ 0 &= 18A+12C \\ 0 &= 3A+2C \end{aligned}$$

So we finally have this system:

$$\begin{aligned} C &= 2A \\ 3A+2C &= 0 \end{aligned}$$

we solve the system and obtain $A=C=0$. Thus the decomposition of a_n is as follows:

$$a_n = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

Thus the sequence of partial sums is as follows:

$$\begin{aligned} s_1 &= a_1 = \frac{1}{1^2} - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} \\ s_2 &= a_1 + a_2 = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) = 1 - \frac{1}{9} = \frac{8}{9} \\ s_3 &= a_1 + a_2 + a_3 = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) = 1 - \frac{1}{16} = \frac{15}{16} \\ s_n &= a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \\ &= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \ldots + \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ &= 1 - \frac{1}{(n+1)^2} \end{aligned}$$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by integral test

4. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n+3^n}{4^n}$ diverges because

$$\sum_{n=0}^{\infty} \frac{5^n+3^n}{4^n} = \sum_{n=0}^{\infty} \frac{5^n}{4^n} + \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$$

and the first geometric series diverges

5. $\displaystyle\sum_{n=0}^{\infty} \sqrt{n}$ diverges since a_n does not approach 0

6. $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{n}$ diverges since a_n does not approach 0

7. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n^2}$ converges since it is a constant times $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges by the integral test:

The function $f(x) = \dfrac{\ln x}{x}$ is continuous and positive on $[2, \infty)$. We may apply the integral test if we also show that f is decreasing there. $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right) = \dfrac{1}{x^2}(1 - \ln x)$ which is negative on $[3, \infty)$. So after $n = 3$, the conditions hold. $\displaystyle\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 + C$. The improper integral $\displaystyle\int_3^{\infty} \frac{\ln x}{x} dx$ diverges and therefore so does the series.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges by the integral test

10. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{e^n}$ converges, a geometric sequence with $a = \dfrac{2}{e}$ and $r = \dfrac{1}{e}$ with $0 < \dfrac{1}{e} < 1$

11. $\displaystyle\sum_{n=1}^{\infty} \operatorname{sech} n$ converges by the integral test

$$\frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right) = 2\frac{d}{dx}\left(e^x + e^{-x}\right)^{-1} = 2(-1)\left(e^x + e^{-x}\right)^{-2}\left(e^x - e^{-x}\right)$$

$$\int \frac{2}{e^x + e^{-x}} dx = 2\arctan(e^x) + C \qquad \int_0^{\infty} \frac{2}{e^x + e^{-x}} dx = \frac{\pi}{2}$$

12. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges

We decompose a_n using partial fractions:

$$\frac{n+2}{n^2-n} = \frac{3}{n-1} - \frac{2}{n}$$

Then s_n becomes $s_n = 2 + \left(\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots\right)$

$$\begin{aligned} s_n &= \frac{3}{1} \underbrace{-\frac{2}{2}+\frac{3}{2}} \underbrace{-\frac{2}{3}+\frac{3}{3}} \underbrace{-\frac{2}{4}+\frac{3}{4}} -\frac{2}{5}+ \ldots\ldots \underbrace{-\frac{2}{n-1}+\frac{3}{n-1}} -\frac{2}{n} \\ &= 3+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\frac{1}{n-1}\right)-\frac{2}{n} \end{aligned}$$

This shows that the sequece of partial sums is not bounded from above, thus the series diverges.

13. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges by the integral test

14. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the integral test

