Theorem: (The Alternating Series Test or Leibniz's Test) The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4.....$$

converges if all three of the following conditions are satisfied:
1) u_n is positive for all n.
2) u_n are (eventually) non-increasing: $u_n \geq u_{n+1}$ for all $n \geq N$ for some integer N.
3) $\lim_{n\to\infty} u_n = 0$

Proof: Assume that $N = 1$. If n is even, say $n = 2k$ then the partial sum s_n is

$$s_n = (u_1 - u_2) + (u_3 - u_4) + ... + (u_{2k-1} - u_{2k})$$

Since $\{u_n\}$ is non-increasing, each difference $u_j - u_{j+1}$ is non-negative, thus s_n is non-negative and non-decreasing. Let us look at s_n again, but this time as

$$s_n = u_1 - (u_2 - u_3) - (u_4 - u_5) - ... - (u_{2k-2} - u_{2k-1}) - u_{2k}$$

This line shows that $s_n \leq u_1$. Thus the sequence $\{s_n\} = \{s_{2k}\}$ of even partial sums is bounded from above (one upper bound is u_1). Because $\{s_n\} = \{s_{2k}\}$ is a sequence that is non-decreasing and bounded from above, it is convergent. Let us denote the limit by L.

Let us now consider the odd partial sums $\{s_{2k+1}\} = s_{2k} + u_{2k+1}$. Recall that $\lim_{n\to\infty} u_n = 0$. Then

$$\lim_{n\to\infty} s_{2k+1} = \lim_{n\to\infty} s_{2k} + u_{2k+1} = \lim_{n\to\infty} s_{2k} + \lim_{n\to\infty} u_{2k+1} = L + 0 = L$$

This completes our proof.

Theorem: (The Alternating Series Estimation Theorem) If an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies Leibniz's test, then

$$s_n = u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n+1}u_n$$

approximates the sum S of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum S lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $S - s_n$ has the same sign as the first unused term.

Definition: A series $\sum a_n$ **converges absolutely** if the corresponding series of absolute values $\sum |a_n|$ converges.

Definition: A series that converges but does not converge absolutely **converges conditionally**.

Theorem: (The Absolute Convergence Test) If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Proof: Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. Then the series $\sum_{n=1}^{\infty} 2|a_n|$ also converges. Consider now the series $\sum_{n=1}^{\infty} (|a_n| + a_n)$. We claim that this series converges as well, by the comparison test: for all $n \in \mathbb{N}$,

$$0 \le |a_n| + a_n \le 2|a_n|$$

So $\sum_{n=1}^{\infty} (|a_n| + a_n)$ also converges. Now consider the difference $\sum_{n=1}^{\infty} (|a_n| + a_n) - |a_n|$. Since this is the difference of two convergent series, it also converges. This completes our proof.

Theorem: (The Rearrangement Theorem for Absolutely Convergent Series) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, b_3, \ldots, b_n, \ldots\ldots$ is any rearrangment of the sequence $\{a_n\}$ then $\sum_{n=1}^{\infty} b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

Note that this is not true for conditionally convergent series.

Sample Problems

1. Determine convergence or divergence of the alternative series.

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$

c) $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n^2-1}$

d) $\sum_{n=1}^{\infty} (-1)^{n} \frac{10^n}{(n+1)!}$

e) $\sum_{n=1}^{\infty} (-1)^{n} \ln\left(1+\frac{1}{n}\right)$

f) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tan^{-1} n}{n^2+1}$

g) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$

h) $\sum_{n=1}^{\infty} (-2)^{-n}$

2. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

c) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

d) $\sum_{n=2}^{\infty} (-1)^{n} \frac{1}{n\ln n}$

e) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$

f) $\sum_{n=1}^{\infty} \frac{(-1)^{n}(n!)^2 3^n}{(2n+1)!}$

f) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$

g) $\sum_{n=1}^{\infty} (-1)^{n} \left(\sqrt{n+1}-\sqrt{n}\right)$

Solutions - Sample Problems

1. Determine convergence or divergence of the alternative series.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+5}{n^2+4}$

diverges since a_n fails to approach zero

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2-1}$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{10^n}{(n+1)!}$

converges since $\{a_n\}$ is alternating, decreasing after $N=11$, and approaches zero

e) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

f) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\tan^{-1}n}{n^2+1}$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

g) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$ diverges since a_n fails to approach zero

h) $\displaystyle\sum_{n=1}^{\infty}(-2)^{-n}$ converges since it is a geometric series with $r=-\frac{1}{2}$

2. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$

The series converges absolutely. We prove it using the ratio test

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{100^{n+1}}{(n+1)!}}{\dfrac{100^n}{n!}} = \lim_{n\to\infty}\left(\frac{100^{n+1}}{(n+1)!}\cdot\frac{n!}{100^n}\right) = \lim_{n\to\infty}\left(\frac{100\cdot 100^n}{(n+1)\,n!}\cdot\frac{n!}{100^n}\right)\\ &= \lim_{n\to\infty}\left(\frac{100}{n+1}\right) = 0 < 1\end{aligned}$$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

This series converges conditionally. $\frac{1}{\sqrt{n}}$ is decrasing and approaches zero. However, it is not absolutely convergent, because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the integral test.

c) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

This series converges absolutely because $\sum_{n=1}^{\infty} \left| \frac{\cos n\pi}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges by the integral test

d) $\sum_{n=2}^{\infty} (-1)^{n} \frac{1}{n \ln n}$

This series converges conditionally because it satisfies Leibniz's test. However, the series $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test

e) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n!)^2}{(2n)!}$

This series converges absolutely because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1} (n!)^2}{(2n)!} \right| = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges by the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{\frac{((n+1)\,n!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)^2 (n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{2(n+1)(2n+1)} = \lim_{n\to\infty} \frac{n+1}{2(2n+1)} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)} = \frac{1}{4} < 1$$

f) $\sum_{n=1}^{\infty} \frac{(-1)^{n} (n!)^2 3^n}{(2n+1)!}$ This series converges absolutely because of the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2 3^{n+1}}{(2(n+1)+1)!}}{\frac{(n!)^2 3^n}{(2n+1)!}} = \lim_{n\to\infty} \frac{((n+1)\,n!)^2 3^n \cdot 3}{(2n+3)!} \cdot \frac{(2n+1)!}{(n!)^2 3^n}$$

$$= \lim_{n\to\infty} \frac{3(n+1)^2 (n!)^2 (2n+1)!}{(2n+3)(2n+2)(2n+1)!\,(n!)^2} = \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+3)(2n+2)} = \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+3)\,2(n+1)}$$

$$= \lim_{n\to\infty} \frac{3(n+1)}{2(2n+3)} = \frac{3}{4} < 1$$

f) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$ This series diverges because a_n fails to approach zero.

g) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1}-\sqrt{n}\right)$

This series converges conditionally, because it satisfies Leibniz's test:

$$\begin{aligned}\lim_{n\to\infty}|a_n| &= \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\left(\left(\sqrt{n+1}-\sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0\end{aligned}$$

However, the series $\displaystyle\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ diverges by the comparison test:

$$\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{\sqrt{n+1}+\sqrt{n+1}} = \frac{1}{2\sqrt{n}}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}$ is divergent.