## Sample Problems

1. Approximate the given integrals using a Taylor series.

   a) $\displaystyle\int_0^1 \sin\left(x^2\right) dx$  b) $\displaystyle\int_0^1 e^{-x^2} dx$

2. Compute the given higher derivatives.

   a) Let $f(x) = \tan^{-1} x$. Compute $f^{(100)}(0)$ and $f^{(2013)}(0)$.

   b) Let $f(x) = \ln(1+x)$. Compute $f^{(35)}(0)$ and $f^{(208)}(0)$.

   c) Let $f(x) = x \sin x$. Compute $f^{(50)}(0)$ and $f^{(99)}(0)$.

3. Evaluate the given indeterminates using a Taylor series.

   a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x - \frac{x^2}{2}}{x^4}$  b) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$

4. Find each of the following sums.

   a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$

   b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3}{4}\right)^{2n}}{(2n)!} = 1 - \frac{3^2}{4^2 \cdot 2!} + \frac{3^4}{4^4 \cdot 4!} - \frac{3^6}{4^6 \cdot 6!} + ....$

   c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(\frac{1}{2}\right)^n = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + ......$

   d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + ....$

   e) $2 + 2^2 + \frac{1}{2!} \cdot 2^3 + \frac{1}{3!} \cdot 2^4 + \frac{1}{4!} \cdot 2^5 + \frac{1}{5!} \cdot 2^6 + \frac{1}{6!} 2^7 + .....$

   f) $-1 + 2x - 3x^2 + 4x^3 - 5x^4 + ......$

   g) $-1 + \frac{2}{3} - \frac{3}{9} + \frac{4}{27} - \frac{5}{81} + ......$

   h) $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + ......$

   i) $1 + \frac{1}{4} + \frac{1}{12} + \frac{1}{32} + \frac{1}{80} + \frac{1}{192} + ......$

# Sample Problems - Solutions

1. Approximate the given integrals using a Taylor series.

a) $\displaystyle\int_0^1 \sin\left(x^2\right) dx$

$$\begin{aligned}\sin x &= x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\dots \quad \text{on } \mathbb{R}\\ \sin\left(x^2\right) &= x^2-\frac{1}{6}x^6+\frac{1}{120}x^{10}-\frac{1}{5040}x^{14}+\dots \quad \text{on } \mathbb{R}\end{aligned}$$

$$\begin{aligned}I &= \int_0^1 \sin\left(x^2\right) dx \approx \int_0^1 \left(x^2-\frac{1}{6}x^6+\frac{1}{120}x^{10}-\frac{1}{5040}x^{14}\right) dx = \left(\frac{1}{3}x^3-\frac{1}{42}x^7+\frac{1}{1320}x^{11}-\frac{1}{75\,600}x^{15}\Big|_0^1\right)\\ &= \frac{1}{3}-\frac{1}{42}+\frac{1}{1320}-\frac{1}{75\,600} \approx \boxed{0.310\,268}\end{aligned}$$

b) $\displaystyle\int_0^1 e^{-x^2} dx$

$$\begin{aligned}e^x &= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\dots \quad \text{on } \mathbb{R}\\ e^{-x^2} &= 1-x^2+\frac{1}{2}x^4-\frac{1}{6}x^6+\frac{1}{24}x^8-\frac{1}{120}x^{10}+\frac{1}{720}x^{12}-\dots\end{aligned}$$

$$\begin{aligned}I &= \int_0^1 e^{-x^2} dx \approx \int_0^1 \left(1-x^2+\frac{1}{2}x^4-\frac{1}{6}x^6+\frac{1}{24}x^8-\frac{1}{120}x^{10}+\frac{1}{720}x^{12}\right) dx\\ &= \left(x-\frac{1}{3}x^3+\frac{1}{10}x^5-\frac{1}{42}x^7+\frac{1}{216}x^9-\frac{1}{1320}x^{11}+\frac{1}{9360}x^{13}\right)\Big|_0^1\\ &= \left(1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}-\frac{1}{1320}+\frac{1}{9360}\right) \approx \boxed{0.746\,8}\end{aligned}$$

2. Compute the given higher derivatives.

a) Let $f(x) = \tan^{-1} x$. Compute $f^{(100)}(0)$ and $f^{(2013)}(0)$.

$$\tan^{-1} x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-\dots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \quad \text{on } [-1,1]$$

We know that in the Taylor polynomial

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \Longrightarrow \quad f^{(n)}(0) = n!a_n$$

$$\begin{aligned}f^{(100)}(0) &= 100!a_{100} = 100!\cdot 0 = 0\\ f^{(2013)}(0) &= 2013!a_{2013} = 2013!\left(-\frac{1}{2013}\right) = \boxed{-2012}\end{aligned}$$

b) Let $f(x) = \ln(1+x)$. Compute $f^{(35)}(0)$ and $f^{(208)}(0)$.

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{on } (-1,1]$$

$$a_n = \frac{f^{(n)}(0)}{n!} \implies f^{(n)}(0) = n!a_n$$

$$\begin{aligned} f^{(35)}(0) &= 35!a_{35} = 35!\left(\frac{1}{35}\right) = 34! \\ f^{(208)}(0) &= 208!a_{208} = 208!\left(-\frac{1}{208}\right) = \boxed{-207!} \end{aligned}$$

c) Let $f(x) = x\sin x$. Compute $f^{(50)}(0)$ and $f^{(99)}(0)$.

$$\begin{aligned} \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ x\sin x &= x^2 - \frac{1}{3!}x^4 + \frac{1}{5!}x^6 - \frac{1}{7!}x^8 + \ldots. = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} \end{aligned}$$

$$a_n = \frac{f^{(n)}(0)}{n!} \implies f^{(n)}(0) = n!a_n$$

$$\begin{aligned} f^{(50)}(0) &= 50!a_{50} = 50!\left(-\frac{1}{49!}\right) = \boxed{-50} \\ f^{(99)}(0) &= 99!a_{99} = 99! \cdot 0 = \boxed{0} \end{aligned}$$

3. Evaluate each of the given indeterminates.

a) $\displaystyle \lim_{x\to 0} \frac{1 - \cos x - \frac{x^2}{2}}{x^4}$

$$\begin{aligned} L &= \lim_{x\to 0} \frac{1 - \cos x - \frac{x^2}{2}}{x^4} = \lim_{x\to 0} \frac{1 - \frac{x^2}{2} - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 - \ldots..\right)}{x^4} \\ &= \lim_{x\to 0} \frac{1 - \frac{x^2}{2} - 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \frac{1}{40\,320}x^8 + \ldots.}{x^4} \\ &= \lim_{x\to 0} \frac{-\frac{1}{24}x^4 + \frac{1}{720}x^6 - \frac{1}{40\,320}x^8 + \ldots.}{x^4} = \lim_{x\to 0} -\frac{1}{24} + \frac{1}{720}x^2 - \frac{1}{40\,320}x^4 + \ldots. = \boxed{-\frac{1}{24}} \end{aligned}$$

b) $\displaystyle \lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} &= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right) - x + \frac{x^3}{6}}{x^5} = \\ &= \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots}{x^5} = \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x}{7!} + \frac{x^2}{9!} - \ldots\right) = \frac{1}{5!} = \boxed{\frac{1}{120}} \end{aligned}$$

 

4. Find each of the following sums.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... = e$ since it is $f(x) = e^x$, evaluated at $x = 1$.

b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3}{4}\right)^{2n}}{(2n)!} = 1 - \frac{3^2}{4^2 \cdot 2!} + \frac{3^4}{4^4 \cdot 4!} - \frac{3^6}{4^6 \cdot 6!} + ....$

If we replace $\frac{3}{4}$ by $x$, we get $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ which is $\cos x$. So, this series is $f(x) = \cos x$, evaluated at $x = \frac{3}{4}$.

the answer is $\boxed{\cos\left(\frac{3}{4}\right) \approx 0.731\,6889}$

c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(\frac{1}{2}\right)^n = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + .....$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(\frac{1}{2}\right)^n = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + ..... = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + .....$$

If we replace $\frac{1}{2}$ by $x$, we get $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + .....$ which is $\ln(x+1)$ on $(-1, 1]$.

So the answer is $\ln\left(1 + \frac{1}{2}\right) = \boxed{\ln\left(\frac{3}{2}\right)}$

d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + ..$

This is $f(x) = \sin x$, evaluated at $x = \frac{\pi}{3}$. So the answer is $\sin\left(\frac{\pi}{3}\right) = \boxed{\frac{\sqrt{3}}{2}}$.

e) $2 + 2^2 + \frac{1}{2!} \cdot 2^3 + \frac{1}{3!} \cdot 2^4 + \frac{1}{4!} \cdot 2^5 + \frac{1}{5!} \cdot 2^6 + \frac{1}{6!} 2^7 + .....$

Write $x$ instead of 2. Then we have

$$S = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \frac{x^6}{5!} + \frac{x^7}{6!} + .... = .x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + ...\right) = xe^x$$

This this sum is $f(x) = xe^x$, evaluated at $x = 2$. This the answer is $\boxed{2e^2}$.

f) $-1 + 2x - 3x^2 + 4x^3 - 5x^4 + ......$

Suppose that $-1+2x-3x^2+4x^3-5x^4+\ldots\ldots=f(x)$.

$$\begin{aligned}
f(x) &= -1+2x-3x^2+4x^3-5x^4+\ldots\ldots \quad \text{integrate both sides}\\
\int f(x)\,dx &= \int\left(-1+2x-3x^2+4x^3-5x^4+\ldots\ldots\right)dx\\
\int f(x)\,dx &= -x+x^2-x^3+x^4-x^5+\ldots+C\\
\int f(x)\,dx &= 1-x+x^2-x^3+x^4-x^5+\ldots+C_2\\
\int f(x)\,dx &= \frac{1}{1+x}+C \quad \text{differentiate both sides}\\
f(x) &= \frac{d}{dx}\left(\frac{1}{1+x}\right)=\boxed{-\frac{1}{(x+1)^2}}. \quad \text{on } (-1,1)
\end{aligned}$$

g) $-1+\frac{2}{3}-\frac{3}{9}+\frac{4}{27}-\frac{5}{81}+\ldots\ldots$ is the function from the previous problem, evaluated at $x=\frac{1}{3}$ So the answer is $-\frac{1}{\left(\frac{1}{3}+1\right)^2}=\boxed{-\frac{9}{16}}$

h) $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\ldots\ldots$

$$\begin{aligned}
1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\ldots\ldots &= f(x) \quad \text{multiply by } x\\
x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\ldots\ldots &= xf(x) \quad \text{differentiate both sides}\\
1+x^2+x^3+x^4+\ldots. &= \frac{d}{dx}(xf(x))\\
\frac{1}{1-x} &= \frac{d}{dx}(xf(x)) \quad \text{on } (-1,1) \quad \text{integrate}\\
-\ln(1-x) &= xf(x) \quad \text{divide by } x\\
f(x) &= \boxed{\frac{-\ln(1-x)}{x}} \quad \text{on } [-1,1)
\end{aligned}$$

i) $1+\frac{1}{4}+\frac{1}{12}+\frac{1}{32}+\frac{1}{80}+\frac{1}{192}+\ldots\ldots$

$$\begin{aligned}
S &= 1+\frac{1}{4}+\frac{1}{12}+\frac{1}{32}+\frac{1}{80}+\frac{1}{192}+\ldots\ldots=1+\frac{\left(\frac{1}{2}\right)}{2}+\frac{\left(\frac{1}{2}\right)^2}{3}+\frac{\left(\frac{1}{2}\right)^3}{4}+\frac{\left(\frac{1}{2}\right)^4}{5}+\frac{\left(\frac{1}{2}\right)^5}{6}+\ldots\ldots\\
&= 1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\ldots., \text{ evaluated at } x=\frac{1}{2}\\
&= \frac{-\ln(1-x)}{x}, \text{ evaluated at } x=\frac{1}{2}\\
&= \frac{-\ln\left(1-\frac{1}{2}\right)}{\frac{1}{2}}=-2\ln\left(\frac{1}{2}\right)=\boxed{2\ln 2}
\end{aligned}$$