Definition: Let f be a function with derivatives of orders $1, 2, ..., n$ in some interval containing a as an interior point. Then the **Taylor polynomial of order n generated by f at $x = a$** is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example 1. Find the Taylor polynomial of order 6 generated by $f(x) = e^x$ at $x = 0$.

$$P_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

Definition: Let f be a function with derivatives of all orders throughout an interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + ...$$

the Taylor series generated by f at $x = 0$.

Example 2. Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where does the series converge to $\frac{1}{x}$?

$$f(x) = x^{-1} \qquad f'(x) = -x^{-2} \qquad f''(x) = 2x^{-3} \qquad f^{(3)}(x) = -6x^{-4} \qquad f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$$

$$f(2) = 2^{-1} = \frac{1}{2} \qquad f'(2) = -2^{-2} = -\frac{1}{4} \qquad f''(2) = 2 \cdot 2^{-3} = \frac{2}{8} = \frac{1}{4} \qquad f^{(3)}(2) = -6 \cdot 2^{-4} = -\frac{3}{8}$$

The Taylor series is

$$\begin{aligned} & f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + ... + \frac{f^{(n)}(2)}{n!}(x-2)^n \\ = & \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 + ... + (-1)^n \frac{1}{2^{n+1}}(x-2)^n \end{aligned}$$

Since this is a geometrical series with first term $\frac{1}{2}$ and common ratio $r = -\frac{x-2}{2}$, it converges absolutely for $\left|\frac{x-2}{2}\right| < 1$ and its sum is $\dfrac{\frac{1}{2}}{1 + \frac{x-2}{2}} = \frac{1}{2 + x - 2} = \frac{1}{x}$.

It is easy to determine radius of convergence but unfortunately, when a Taylor series generated by f converges, it may or may not converge to f. One example is the function

$$f(x) = \begin{cases} 0 & \text{if } \; x = 0 \\ e^{-1/x^2} & \text{if } \; x \neq 0 \end{cases}$$

This function has derivatoves of all orders at $x = 0$ and $f^{(n)}(x) = 0$ for all n. Thus the Taylor series at $x = 0$ is

$$0 + 0 \cdot x + 0 \cdot x^2 + ... + 0 \cdot x^n + ...$$

which is clearly convergent for all x but does not converge to $f(x)$.

Theorem: (Taylor's Theorem) Suppose that $f, f', f'', ..., f^{(n)}$ are continuous on an open interval I containing a. Then for all positive integer n and all x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$\text{where} \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x.$$

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ converges to f on I and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example 3. Find all values of x for which the Taylor series generated by $f(x) = e^x$ converges to its generating function.

By Taylor's theorem, $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + R_n(x)$$

where the remainder term is

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \qquad \text{for some } c \text{ between } 0 \text{ and } x \\ R_n(x) &= \frac{e^c}{(n+1)!}x^{n+1} \end{aligned}$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{e^c}{(n+1)!}x^{n+1} = e^c \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

and so the Taylor series converges to e^x for all x.

Important Taylor series:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \qquad \text{on } \mathbb{R} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \qquad \text{on } \mathbb{R} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \qquad \text{on } \mathbb{R} \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + ... \qquad \text{on } (-1,1) \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 + ... \qquad \text{on } (-1,1) \\ \ln(x+1) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \qquad \text{on } (-1,1] \\ \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \qquad \text{on } [-1,1] \end{aligned}$$

Sample Problems

1. Find the Taylor series generated by each of the following functions.

   a) $f(x) = x \sin x$ b) $f(x) = e^{3x}$ c) $f(x) = x^2 \tan^{-1} x$ d) $f(x) = \sinh x$

2. Use a Taylor series to determine the sum of each of the following series.

   a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} ..... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

   b) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - .... = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$

   c) $\frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1} (2n+1)!}$

   d) $-1 + 2x - 3x^2 + 4x^3 - 5x^4 .... = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$

   e) $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$

3. Use the Taylor series generated by $f(x) = x \tan^{-1} x$ to determine the value of $f^{(208)}(0)$.

4. Consider the Taylor series generated by $f(x) = e^x$.

   a) Let i be the complex number with $i^2 = -1$.Simplify the expressions i^2, i^3, i^4, i^5, i^6, ...

   b) Write the Taylor series generated by $f(x) = e^x$ and evaluate it at $x = i\theta$ to prove that $e^{i\theta} = \cos\theta + i \sin\theta$

   c) Use the result $e^{i\theta} = \cos\theta + i \sin\theta$ to compute $e^{i\pi}$

Solutions - Sample Problems

1. Find the Taylor series generated by each of the following functions.

a) $f(x) = x \sin x$

Solution: We take the Taylor series generated by $\sin x$ and multiply it by x.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \qquad \text{on } \mathbb{R} \\ x \sin x &= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + ... \qquad \text{on } \mathbb{R} \end{aligned}$$

b) $f(x) = e^{3x}$

Solution: We take the Taylor series generated by e^x and substitute $3x$ into it.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{on } \mathbb{R} \\ e^{3x} &= 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \qquad \text{on } \mathbb{R} \\ &= 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 + ... = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n \qquad \text{on } \mathbb{R} \end{aligned}$$

c) $f(x) = x^2 \tan^{-1} x$

Solution: We take the Taylor series generated by $f(x) = \tan^{-1} x$ and multiply it by x^2.

$$\begin{aligned} \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{on } [-1,1] \\ x^2 \tan^{-1} x &= x^3 - \frac{x^5}{3} + \frac{x^7}{5} - \frac{x^9}{7} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1} \qquad \text{on } [-1,1] \end{aligned}$$

d) $f(x) = \sinh x$

Solution: We will find the Taylor series directly from the derivatives $f^{(n)}(0)$. Recall that $a_n = \dfrac{f^{(n)}(0)}{n!}$

$$\begin{aligned} f(x) &= \sinh x \implies f(0) = 0 \implies a_0 = \frac{f(0)}{0!} = \frac{0}{1} = 0 \\ f'(x) &= \cosh x \implies f'(0) = 1 \implies a_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1 \\ f''(x) &= \sinh x \implies f''(0) = 0 \implies a_2 = \frac{f''(0)}{2!} = \frac{0}{2} = 0 \\ f^{(3)}(x) &= \cosh x \implies f^{(3)}(0) = 1 \implies a_3 = \frac{f^{(3)}(0)}{3!} = \frac{1}{6} \\ f^{(4)}(x) &= \sinh x \implies f^{(4)}(0) = 0 \implies a_4 = \frac{f^{(4)}(0)}{4!} = \frac{0}{24} = 0 \end{aligned}$$

and so on, as the derivatives repeat, we have an easy pattern, and so the taylor series for $\sinh x$ is

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$$

2. Use a Taylor series to determine the sum of each of the following series.

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}..... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Solution: We start with the Taylor series for $\ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - .... \qquad \text{on } (-1,1]$$

Let $x = 1$

$$\begin{aligned} \ln(1+1) &= 1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} + \frac{1^5}{5} - ... \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ...... \end{aligned}$$

Thus the sum of the series is $\ln 2$.

b) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - .... = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$

Solution: We start by the Taylor series generated by e^x and evaluate it at $x = -1$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{on } \mathbb{R} \\ e^{-1} &= 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \end{aligned}$$

Thus the sum of the series is $\frac{1}{e}$

c) $\frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1}(2n+1)!}$

Solution: We start by the Taylor series generated by $\sin x$ and evaluate it at $x = \frac{\pi}{3}$.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} \qquad \text{on } \mathbb{R} \\ \sin \frac{\pi}{3} &= \frac{\pi}{3} - \frac{\left(\frac{\pi}{3}\right)^3}{3!} + \frac{\left(\frac{\pi}{3}\right)^5}{5!} - \frac{\left(\frac{\pi}{3}\right)^7}{7!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1}(2n+1)!} \end{aligned}$$

Thus the sum of the series is $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

d) $-1 + 2x - 3x^2 + 4x^3 - 5x^4.... = \sum_{n=1}^{\infty} (-1)^n nx^{n-1}$

Solution: Notice that if we integrate the series term by term, we get

$$-x + x^2 - x^3 + x^4 - x^5.... = \sum_{n=1}^{\infty} (-1)^n x^n$$

This is a geometric series with first element $-x$ and common ratio $-x$. Thus this series converges if $|x| < 1$ to the sum $\frac{-x}{1-(-x)} = -\frac{x}{1+x}$.

$$-x + x^2 - x^3 + x^4 - x^5.... = -\frac{x}{1+x} \qquad \text{on } (-1,1)$$

Differentiating both sides of this we obtain

$$-1 + 2x - 3x^2 + 4x^3 - 5x^4.... = -\frac{1}{(x+1)^2} \qquad \text{on } (-1,1)$$

and so the sum is $-\dfrac{1}{(x+1)^2}$ for all $-1 < x < 1$.

e) $1 + \dfrac{x}{2} + \dfrac{x^2}{3} + \dfrac{x^3}{4} + ... = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1}$

Solution: Notice that when we multiply this series by x and differentiate it, we get

$$\begin{aligned}
f(x) &= 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n+1} \\
xf(x) &= x\left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + ...\right) = x\left(\sum_{n=0}^{\infty} \frac{x^n}{n+1}\right) \\
xf(x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ... = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\
\frac{d}{dx}(xf(x)) &= 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n \\
\frac{d}{dx}(xf(x)) &= \frac{1}{1-x} \quad \text{on} \quad (-1,1) \\
xf(x) &= \int \frac{1}{1-x} dx \quad \text{on} \quad (-1,1) \\
xf(x) &= -\ln(1-x) \quad \text{on} \quad (-1,1) \\
f(x) &= -\frac{1}{x}\ln(1-x) \quad \text{on} \quad (-1,1)
\end{aligned}$$

3. Use the Taylor series generated by $f(x) = x\tan^{-1} x$ to determine the value of $f^{(208)}(0)$.

Solution: We start with the Taylor series generated by $g(x) = \tan^{-1} x$. We then multiply that by x

$$\begin{aligned}
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{on } [-1,1] \\
x\tan^{-1} x &= x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1} \qquad \text{on } [-1,1]
\end{aligned}$$

Let us now focus on the 208 degree term

$$\begin{aligned}
\frac{f^{(208)}(0)}{208!} x^{208} &= (-1)^{103} \frac{x^{208}}{207} \\
\frac{f^{(208)}(0)}{208!} &= -\frac{1}{207} \\
f^{(208)}(0) &= -\frac{208!}{207} = -208 \cdot 206!
\end{aligned}$$

4. Consider the Taylor series generated by $f(x) = e^x$.

a) Let i be the complex number with $i^2 = -1$.Simplify the expressions i^2, i^3, i^4, i^5, i^6, ...

Solution:

$$\begin{aligned} i^3 &= \left(i^2\right) i = (-1)\, i = -i \\ i^4 &= \left(i^3\right) i = (-i)\, i = -i^2 = -(-1) = 1 \\ i^5 &= \left(i^4\right) i = (1)\, i = i \\ i^6 &= \left(i^5\right) i = (i)\, i = i^2 = -1 \end{aligned}$$

In general, if n is an integer, then

$$\begin{aligned} i^{4n} &= \left(i^4\right)^n = 1^n = 1 \\ i^{4n+1} &= i^{4n} \cdot i^1 = \left(i^4\right)^n \cdot i = 1^n \cdot i = 1 \cdot i = i \\ i^{4n+2} &= i^{4n} \cdot i^2 = \left(i^4\right)^n \cdot i^2 = 1^n (-1) = 1(-1) = -1 \\ i^{4n+3} &= i^{4n} \cdot i^3 = \left(i^4\right)^n \cdot i^3 = 1^n \cdot i^3 = 1(-i) = -i \end{aligned}$$

b) Write the Taylor series generated by $f(x) = e^x$ and evaluate it at $x = i\theta$ to prove that $e^{i\theta} = \cos\theta + i\sin\theta$

Solution:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} \\ e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} \end{aligned}$$

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + ...\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n\, i\frac{\theta^{2n+1}}{(2n+1)!} \\ e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\ e^{i\theta} &= \cos\theta + i\sin\theta \end{aligned}$$

c) Use the result $e^{i\theta} = \cos\theta + i\sin\theta$ to compute $e^{i\pi}$

Solution: $e^{i\pi} = \cos\pi + i\sin\pi = -1$

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