Definitions:
(*Interior point*) A function $y = f(x)$ is continuous at an interior point c of its domain if $\lim_{x \to c} f(x) = f(c)$.
(*Endpoint*) A function $y = f(x)$ is continuous at a left endpoint a or is continuous at a right endpoint b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \text{ respectively.}$$

A function f is said to be continuous on an open interval (a, b) if it is continuous at every point inside the interval. A function f is said to be continuous on a closed interval $[a, b]$ if it is continuous at every point in the interval as described above. (End-points of the interval require only one-sided limits.)

We can use operations on functions to create new functions. For example, given the functions f and g, we can define new functions as follows. $f + g$, $f - g$, fg, and $\frac{f}{g}$ are defined as.

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) && \text{for all } x \text{ in the domains of both } f \text{ and } g \\
(f-g)(x) &= f(x) - g(x) && \text{for all } x \text{ in the domains of both } f \text{ and } g \\
(fg)(x) &= f(x)\, g(x) && \text{for all } x \text{ in the domains of both } f \text{ and } g \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} && \text{for all } x \text{ in the domains of both } f \text{ and } g \text{ and where } g(x) \neq 0
\end{aligned}$$

Another operation on functions is to compose them, that is to apply the rule for first one function and then the other. The new function that is created is denoted by $f \circ g$ and is defined by

$$(f \circ g)(x) = f(g(x)) \quad \text{where } x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f$$

Notice that composing functions is not commutative, i.e. very often $f \circ g \neq g \circ f$.

Theorem: If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Theorem: If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g\left(\lim_{x \to c} f(x)\right)$$

Continuous functions have very nice properties and we easily visualize continuous functions. However, many, many functions are not continuous and their study might be more difficult. Functions that are continuous on a closed interval have especially nice properties.

Theorem: If f is continuous on a closed interval $[a, b]$, where $f(a)$ is negative and $f(b)$ is positive, then there exists c in (a, b) such that $f(c) = 0$.

Theorem: (**The Intemediate value Theorem for Continuous Functions**) If f is continuous on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Example 1: Suppose that f is a function defined as $f(x) = \begin{cases} mx - 10 & \text{if } x < -2 \\ x^2 + 9x - 8 & \text{if } x \geq -2 \end{cases}$. Find the value of m if we know that f is continuous everywhere.

Solution: If $x < -2$, then the function is continuous for all x. Similarly, f is also continuous on all x with $x \geq -2$. The only questionable point is at $x = -2$. For a continuous function, we need the left limit and the right limit to exist and have the same value.

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^+} f(x) \\ \lim_{x \to -2^-} (mx - 10) &= \lim_{x \to -2^+} \left(x^2 + 9x - 8\right) \end{aligned}$$

By the various properties of limits, this equation can be simplified as follows:

$$\begin{aligned} m(-2) - 10 &= (-2)^2 + 9(-2) - 8 \\ -2m - 10 &= -22 \\ -2m &= -12 \\ m &= 6 \end{aligned}$$

And so $m = 6$ is the value for which f is continuous on the entire number line.

Practice Problems

1. Suppose that f is a function defined as $f(x) = \begin{cases} mx - 13 & \text{if } x < -10 \\ x^2 + 5x - 3 & \text{if } x \geq -10 \end{cases}$. Find the value of m if we know that f is continuous everywhere.

2. Suppose that f is a function defined as $f(x) = \begin{cases} 8x - 4 & \text{if } x \leq 4 \\ -2x + b & \text{if } x > 4 \end{cases}$. Find the value of b if we know that f is continuous everywhere.

3. Suppose that f is a function defined as $f(x) = \begin{cases} mx - 11 & \text{if } x < -6 \\ x^2 + 4x - 5 & \text{if } x \geq -6 \end{cases}$. Find the value of m if we know that f is continuous everywhere.

4. Suppose that f is a function defined as $f(x) = \begin{cases} 2x + b & \text{if } x < 7 \\ \sqrt{x + 2} & \text{if } x \geq 7 \end{cases}$. Find the value of b if we know that f is continuous everywhere.

Answers - Practice Problems

1.) -6 2.) 36 3.) -3 4) -11

