Definition: A set S is **bounded from above** if there exists a real number U such that for all x in S, $x \leq U$.

Definition: A set S is **bounded from below** if there exists a real number L such that for all x in S, $x \geq L$.

Definition: A set S is **bounded** if it is bounded from above and from below.

Axiom: (**Least Upper Bound Property**) Every non-empty, set S of real numbers has the following property: if S is bounded from above, then there exists a least upper bound for S.

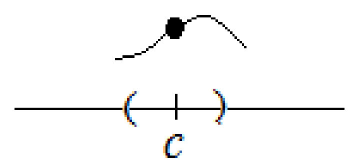
The least upper bound is also called supremum. The least upper bound property is a very fundamental one: it is actually the single axiom that distinguishes the set of rational numbers from the set of real numbers. Rational numbers do not have this propoerty. This property is also the key ingredient in proving the Intermediate Value Theorem.

Theorem: (**The Intermediate Value Theorem**) If f is continuous on a closed interval $[a, b]$ and if $f(a) < 0$ and $f(b) > 0$, then there exists c in (a, b) so that $f(c) = 0$.

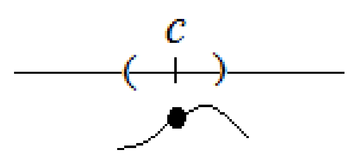
Proof: Suppose that the conditions hold. Define $S = \{x : a \leq x \leq b \text{ and } f(x) < 0\}$. This set is non-empty because a is an element of it. This set is also bounded from above, because for all x in S, $x \leq b$ and so b is an upper bound for S. By the least upper bound property, S has a least upper bound. Let us detone it by c. Since c is the least upper bound for S and b is an upper bound, we also have that $c \leq b$. Since a is in S and c is an upper bound, we also have that $a \leq c$. Thus c is in the interval $[a, b]$. We will prove that $f(c) = 0$.

We will prove that $f(c) = 0$ by showing that $f(c)$ cannot be positive or negative.

Suppose first that $f(c)$ is positive. Since f is continuous at c, that means that f is positive on some open interval containing c. That means that c is not the least upper bound for S, because any number in that interval, to the left of c is also an upper bound for S. That is impossible and so $f(c)$ cannot be positive.



Suppose now that $f(c)$ is negative. Since f is continuous at c, that means that f is negative on some open interval containing c. That means that c is not an upper bound for S, because a number in that interval, to the right of c is also an element of S. That is impossible and so $f(c)$ cannot be negative.



So $f(c) = 0$ which completes our proof. ■

Theorem: Suppose that f and g are functions that are continuous at $x = c$. Then:

1) $f + g$ is continuous at c.
2) fg is continuous at c
3) $f - g$ is continuous at c
4) If $g(c) \neq 0$, then $\dfrac{f}{g}$ is continuous at c
5) If g is continuos at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Theorem: (**The Intermediate Value Theorem for Continuous Functions**) If f is continuous on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Proof: Apply the Intermediate Value Theorem for $h(x) = f(x) - y_0$.■

Applications of the Intermediate Value Theorem

1. Prove that the function $f(x) = x^5 - 3x^4 + 8x^3 - x - 2$ has at least one zero in the interval $[0, 1]$.
2. Prove that the function $f(x) = 6x^4 + x^3 - 25x^2 - 4x + 4$ has at least two zeroes in the interval $[-1, 1]$.
3. Prove that all polynomials with an odd degree have at least one zero.

Solutions

1. f is continuous on $[0, 1]$, $f(0) = -2$, and $f(1) = 3$. By the Intermediate Value Theorem, f has a zero in $[0, 1]$.
2. f is continuous on $[-1, 1]$. $f(-1) = -12$ and $f(0) = 4$. By the Intermediate Value Theorem, f has a zero in $[-1, 0]$. Also, $f(0) = 4$ and $f(1) = -18$. By the Intermediate Value Theorem, f has a zero in $[0, 1]$.
3. Suppose that f is an odd degree polynomial with a positive leading coefficient. Then $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$. Then there exist sufficiently a large negative value of x (denoted by a) so that $f(a)$ is negative. Also, there exist sufficiently a large positive value of x (denoted by b) so that $f(b)$ is positive. Since f is a polynomial, it is continuous on $\mathbb{R}$ and thus on $[a, b]$. By the Intermediate Value Theorem, f must have a zero in the interval $[a, b]$. The proof goes similarly for functions with negative leading coefficients.