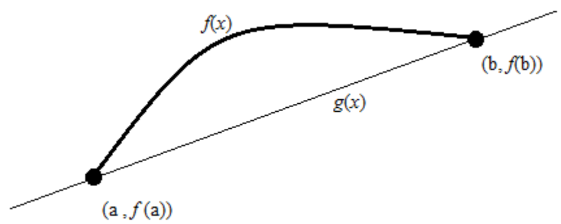
Theorem: (Rolle) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists c in (a, b) with $f'(c) = 0$.

Proof: Suppos that that f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. By the extreme value theorem, f has an absolute maximum and minimum on $[a, b]$. If that maximum or minimum is at an interior point c, then $f'(c) = 0$. If both the maximum and minimum are at an endpoint, then $f(a) = f(b)$ means that the function is constant and so at all interior point c we have $f'(c) = 0$.

Theorem: (**Mean Value Theorem**) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists c in (a, b) with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof: Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Define g a linear function to be the line connecting $f(a)$ and $f(b)$.



The equation of this line is

$$g(x) = m(x-a) + f(a) \qquad \text{where } m = \frac{f(b)-f(a)}{b-a} \text{ and so}$$
$$g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

Clearly g is differentiable on $\mathbb{R}$. Now define the difference function h between f and g.

$$h(x) = f(x) - g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]$$
$$h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$

We will be able to apply Rolle's Theorem to this function. Since both f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h. Furthermore, $h(a) = h(b) = 0$.

$$h(a) = f(a) - g(a) = f(a) - \left(\frac{f(b)-f(a)}{b-a}(a-a) + f(a)\right) = f(a) - f(a) = 0 \text{ and}$$
$$h(b) = f(b) - g(b) = f(b) - \left(\frac{f(b)-f(a)}{b-a}(b-a) + f(a)\right) = f(b) - f(b) = 0$$

and so the conditions hold for Rolle's Theorem. By this theorem, there exists c in (a, b) so that $h'(c) = 0$. We differentiate h

$$h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$
$$= f(x) - \frac{f(b)-f(a)}{b-a} \cdot x + \frac{f(b)-f(a)}{b-a} \cdot a - f(a)$$

$$\begin{aligned}
h'(x) &= f'(x) - \frac{f(b)-f(a)}{b-a} \quad \text{and for some } c \text{ in } (a,b), \quad h'(c)=0 \\
0 &= h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \\
0 &= f'(c) - \frac{f(b)-f(a)}{b-a} \\
f'(c) &= \frac{f(b)-f(a)}{b-a}
\end{aligned}$$

which completes our proof.

Corollary 1. If $f'(x) = 0$ at each point of an open interval (a,b), then $f(x) = C$ for all x in (a,b) for some constant C.

Proof: Suppose that $f'(x) = 0$ at each point of an open interval (a,b). Let x_1 and x_2 be any two different points in (a,b). Since f is differentiable, it is also continuous on $[x_1, x_2]$. We will apply the Mean Value Theorem. The slope $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ is achieved as a derivative somewhere in the interval $[x_1, x_2]$. There exists c between x_1 and x_2 so that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

By the hypotheses, $f'(c) = 0$. This gives us

$$\begin{aligned}
\frac{f(x_2)-f(x_1)}{x_2-x_1} &= 0 \\
f(x_2)-f(x_1) &= 0 \\
f(x_2) &= f(x_1)
\end{aligned}$$

and so f is constant on (a,b).

Corollary 2. If $f'(x) = g'(x)$ at each point x in an open interval (a,b), then there exists a constant C such that $f(x) = g(x) + C$ for all x in (a,b). That is, $f - g$ is constant on (a,b).

Proof: Suppose that $f'(x) = g'(x)$ on (a,b). Define $h(x) = f(x) - g(x)$ on (a,b). Then h is differentiable and

$$h'(x) = (f(x) - g(x))' = f'(x) - g'(x) = 0$$

By the previous corollary, $h'(x) = 0$ on (a,b) inplies that $h(x) = C$ on (a,b) for some constant C. Thus

$$\begin{aligned}
h(x) &= C \quad \text{on } (a,b) \\
f(x) - g(x) &= C \\
f(x) &= g(x) + C \quad \text{on } (a,b)
\end{aligned}$$

which completes our proof.

Practice Problems

1. Find all values of c that satisfies the conclusion of the Mean Value Theorem.

a) $f(x) = 5x^2 - 3x + 8$ on $[-1, 4]$

b) $f(x) = x^2 + 2x - 1$ on $[0, 1]$

c) $f(x) = x^{2/3}$ on $[0, 1]$

d) $f(x) = x + \dfrac{1}{x}$ on $\left[\dfrac{1}{2}, 2\right]$

e) $f(x) = \ln x$ on $[1, 10]$

f) $f(x) = \sqrt{x-1}$ on $[1, 3]$

g) $f(x) = 2x^3 - 5x + 7$ on $[-2, 2]$

2. Suppose that $3 \le f'(x) \le 5$ for all values of x. Show that $18 \le f(8) - f(2) \le 30$.

Answers - Practice Problems

1. a) $\dfrac{3}{2}$ b) $\dfrac{1}{2}$ c) $\dfrac{8}{27}$ d) 1 e) $\dfrac{9}{\ln 10}$ f) $\dfrac{3}{2}$ g) $\pm\dfrac{2\sqrt{3}}{3}$

2. see solutions

Solutions - Practice Problems

1. Find all values of c that satisfies the conclusion of the Mean Value Theorem.

a) $f(x) = 5x^2 - 3x + 8$ on $[-1, 4]$

Solution: We first evaluate the function at the endpoints of the interval. $f(-1) = 16$ and $f(4) = 76$. The slope of secant line connecting the two points is $\dfrac{76 - 16}{4 - (-1)} = 12$. So we are looking for all values of c for which $f'(c) = 12$. $f'(x) = 10x - 3$, so we solve $10x - 3 = 12$ and obtain $x = \dfrac{3}{2}$

b) $f(x) = x^2 + 2x - 1$ on $[0, 1]$

Solution: We first evaluate the function at the endpoints of the interval. $f(0) = -1$ and $f(1) = 2$. The slope of secant line connecting the two points is $\dfrac{2 - (-1)}{1 - 0} = 3$. So we are looking for all values of c for which $f'(c) = 3$. $f'(x) = 2x + 2$, so we solve $2x + 2 = 3$ and obtain $x = \dfrac{1}{2}$

c) $f(x) = x^{2/3}$ on $[0, 1]$

Solution: We first evaluate the function at the endpoints of the interval. $f(0) = 0$ and $f(1) = 1$. The slope of secant line connecting the two points is $\dfrac{1 - 0}{1 - 0} = 1$. So we are looking for all values of c for which $f'(c) = 1$. $f'(x) = \dfrac{2}{3\sqrt[3]{x}}$, so we solve $\dfrac{2}{3\sqrt[3]{x}} = 1$ and obtain $x = \dfrac{8}{27}$.

d) $f(x) = x + \dfrac{1}{x}$ on $\left[\dfrac{1}{2}, 2\right]$

Solution: We first evaluate the function at the endpoints of the interval. $f\left(\dfrac{1}{2}\right) = \dfrac{5}{2}$ and $f(2) = \dfrac{5}{2}$. The slope of secant line connecting the two points is 0. So we are looking for all values of c for which $f'(c) = 0$. $f'(x) = 1 - \dfrac{1}{x^2}$, so we solve $1 - \dfrac{1}{x^2} = 0$ and obtain $x = \pm 1$. Recall that we are looking for numbers tha fall in the interval $\left[\dfrac{1}{2}, 2\right]$. That rules out -1 and so the only solution is $x = 1$.

e) $f(x) = \ln x$ on $[1, 10]$

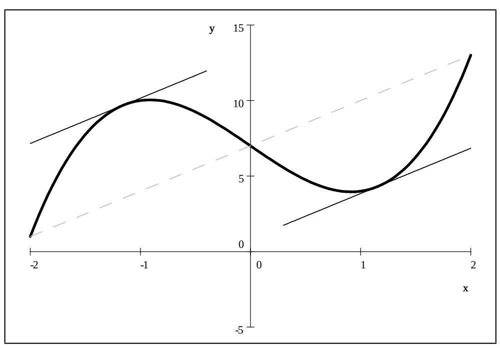
Solution: We first evaluate the function at the endpoints of the interval. $f(1) = 0$ and $f(10) = \ln 10$. The slope of secant line connecting the two points is $\dfrac{\ln 10 - 0}{10 - 1} = \dfrac{\ln 10}{9}$. So we are looking for all values of c for which $f'(c) = \dfrac{\ln 10}{9}$. $f'(x) = \dfrac{1}{x}$, so we solve $\dfrac{1}{x} = \dfrac{\ln 10}{9}$ and obtain $x = \dfrac{9}{\ln 10}$. This number is approximately $3.908\,65$ and so it is in the iterval $[1, 10]$.

f) $f(x) = \sqrt{x-1}$ on $[1, 3]$

Solution: We first evaluate the function at the endpoints of the interval. $f(1) = 0$ and $f(3) = \sqrt{2}$. The slope of secant line connecting the two points is $\dfrac{\sqrt{2} - 0}{3 - 1} = \dfrac{\sqrt{2}}{2}$. So we are looking for all values of c for which $f'(c) = \dfrac{\sqrt{2}}{2}$. $f'(x) = \dfrac{1}{2\sqrt{x-1}}$ =, so we solve $\dfrac{1}{2\sqrt{x-1}} = \dfrac{\sqrt{2}}{2}$, and obtain $x = \dfrac{3}{2}$. $(x) = \dfrac{1}{2\sqrt{x-1}} = \dfrac{1}{\sqrt{2}}$

g) $f(x) = 2x^3 - 5x + 7$ on $[-2, 2]$

Solution: We first evaluate the function at the endpoints of the interval. $f(-2) = 1$ and $f(2) = 13$. The slope of secant line connecting the two points is $\dfrac{13 - 1}{2 - (-2)} = 3$. So we are looking for all values of c for which $f'(c) = 3$. $f'(x) = 6x^2 - 5$, so we solve $6x^2 - 5 = 3$, and obtain $x = \pm\dfrac{2\sqrt{3}}{3}$. Both of these numbers satisfy the conclusion of the Mean Value Theorem.



2. Suppose that $3 \le f'(x) \le 5$ for all values of x. Show that $18 \le f(8) - f(2) \le 30$.

   Proof: By the mean value theorem, there exists c in $[2, 8]$ with $f'(c) = \dfrac{f(8) - f(2)}{8 - 2} = \dfrac{f(8) - f(2)}{6}$.

   Since $3 \le f'(x) \le 5$ for all values of x, we have that

$$\begin{aligned} 3 &\le \frac{f(8) - f(2)}{6} \le 5 \qquad \text{multiply by 6} \\ 18 &\le f(8) - f(2) \le 30 \end{aligned}$$

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