Theorem: (**The Intermediate Value Theorem**) If f is continuous on a closed interval $[a,b]$ and if $f(a)<0$ and $f(b)>0$, then there exists c in (a,b) so that $f(c)=0$.

Theorem: (**The Intermediate value Theorem for Continuous Functions**) If f is continuous on a closed interval $[a,b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0=f(c)$ for some c in $[a,b]$.

Applications of the Intermediate Value Theorem

1. Prove that the function $f(x)=x^5-3x^4+8x^3-x-2$ has at least one zero in the interval $[0,1]$.
2. Prove that the function $f(x)=6x^4+x^3-25x^2-4x+4$ has at least two zeroes in the interval $[-1,1]$.
3. Prove that all polynomials with an odd degree have at least one zero.

Solutions

1. f is continuous on $[0,1]$, $f(0)=-2$, and $f(1)=3$. By the Intermediate Value Theorem, f has a zero in $[0,1]$.
2. f is continuous on $[-1,1]$. $f(-1)=-12$ and $f(0)=4$. By the Intermediate Value Theorem, f has a zero in $[-1,0]$. Also, $f(0)=4$ and $f(1)=-18$. By the Intermediate Value Theorem, f has a zero in $[0,1]$.
3. Suppose that f is an odd degree polynomial with a positive leading coefficient. Then $\lim\limits_{x\to-\infty} f(x)=-\infty$ and $\lim\limits_{x\to-\infty} f(x)=-\infty$. Then there exist sufficiently a large negative value of x (denoted by a) so that $f(a)$ is negative. Also, there exist sufficiently a large positive value of x (denoted by b) so that $f(b)$ is positive. Since f is a polynomial, it is continuous on $\mathbb{R}$ and thus on $[a,b]$. By the Intermediate Value Theorem, f must have a zero in the interval $[a,b]$. The proof goes similarly for functions with negative leading coefficients.

Maximum-Minimum Theorems

Definition: Let f be a function with domain D. Then f has a **relative maximum value** at a point c if $f(x) \leq f(c)$ for all x in D lying in some open interval containing c. A function f has a **relative minimum value** at a point c if $f(x) \geq f(c)$ for all x in D lying in some open interval containing c.

Theorem: (First Derivative Theorem for Local Extreme Values) If f is has a relative maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof: Suppose that f is differentiable at c and f has a local maximum at c. Then $f'(c) = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ exists and is a two-sided limit. For a sufficiently small positive value of h, $f(c+h)$ exists and $f(c+h) \leq f(c)$ since f has a local maximum value at c. Then $f(c+h) - f(c) \leq 0$. Divide that by positive h and get that $\dfrac{f(c+h) - f(c)}{h} \leq 0$ and so

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

Now let h be a very small negative number. Then by the same argument, $f(c+h) \leq f(c)$. Divide that by a negative h and get that $\dfrac{f(c+h) - f(c)}{h} \geq 0$ and so

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

For the two-sided limit $f'(c)$ to exists, we must have

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

Since one side is less than or equal to zero and the other is greater than or equal to zero, they both must be zero.

Definition: A **critical number** of a function f is a number c in its domain such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem: (Fermat) If f has a local maximum or minimum at c, then c is a critical number of f.

Definition: Let f be a function with domain D. Then f has an **absolute maximum value** on D at a point c if $f(x) \leq f(c)$ for all x in D and an **absolute minimum value** on D at a point c if $f(x) \geq f(c)$ for all x in D.

Theorem: (**Extreme Value Theorem**) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Closed interval method: To find absolute extrema of a continuous function f on a closed interval $[a, b]$.
1) Find the values of f at the critical numbers of f in $[a, b]$.
2) Find the values of f at the endpoints of the interval.
3) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

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