Definition: A set $S$ is **bounded from above** if there exists a real number $U$ such that for all $x$ in $S$, $x \leq U$.

Definition: A set $S$ is **bounded from below** if there exists a real number $L$ such that for all $x$ in $S$, $x \geq L$.

Definition: A set $S$ is **bounded** if it is bounded from above and from below.

Axiom: (**Least Upper Bound Property**) Every non-empty, set $S$ of real numbers has the following property: if $S$ is bounded from above, then there exists a least upper bound for $S$.

The least upper bound is also called supremum. The least upper bound property (also called the completeness property) is a very fundamental one: it is actually the single axiom that distinguishes the set of rational numbers from the set of real numbers. Rational numbers do not have this property, but real numbers do. In calculus, the least upper bound property is a key ingredient in proving very important theorems later.

The following is the actual definition of a (finite) limit of a function $f$ at a number $c$. In this course, we will not use this definiton.

Definition: Suppose that $f$ is a function and $c$, $L$ are real numbers. We say that $\lim\limits_{x\to c} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \neq c$ with $|x-c| < \delta$, we also have that ($f$ is defined and) $|f(x) - L| < \varepsilon$.

The following definition is the one we will use.

Definition: If the left-hand side limit and the right-hand side limit both exist (and are finite) and are equal, we say that $\lim\limits_{x\to c} f(x) = L$.

Definition: (Continuity at a point) A function $y = f(x)$ is continuous at a number $c$ of its domain if the two-sided limit exists and $\lim\limits_{x\to c} f(x) = f(c)$.

Definition: (Continuity on an interval)
(*Open Interval*) A function $y = f(x)$ is continuous on an interval $(a,b)$ if it is continuous at every $c$ in $(a,b)$.
(*Closed Interval*) A function $y = f(x)$ is continuous on an interval $[a,b]$ if it is continuous at every $c$ in $(a,b)$ and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b)\,, \text{ respectively.}$$

End-points of the interval require only one-sided limits.

Another way to express continuity is to say that $\lim\limits_{h\to 0} f(x+h) = f(x)$. Another alternative statement of continuity is $\lim\limits_{x\to c} f(x) = f\left(\lim\limits_{x\to c} x\right)$, so that there is a commutativity between taking the limit and taking the function values.

Theorem: Suppose that $f$ and $g$ are functions that are continuous at $x = c$. Then:

1) $f+g$ is continuous at $c$.
2) $fg$ is continuous at $c$
3) $f-g$ is continuous at $c$
4) If $g(c) \neq 0$, then $\dfrac{f}{g}$ is continuous at $c$
5) If $g$ is continuos at $c$ and $f$ is continuous at $g(c)$, then $f \circ g$ is continuous at $c$.

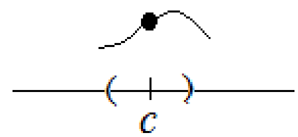
These properties can be proved using the properties of limits and the definitions of the functions $f+g$, $fg$, $f-g$, $\dfrac{f}{g}$ and $f \circ g$.

Theorem: (**The Intermediate Value Theorem**) If $f$ is continuous on a closed interval $[a,b]$ and if $f(a) < 0$ and $f(b) > 0$, then there exists $c$ in $(a,b)$ so that $f(c) = 0$.

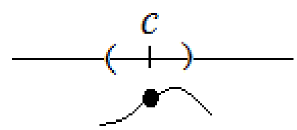
Proof: Suppose that the conditions hold. Define $S = \{x : a \leq x \leq b \text{ and } f(x) < 0\}$. This set is non-empty because $a$ is an element of it. This set is also bounded from above, because for all $x$ in $S$, $x \leq b$ and so $b$ is an upper bound for $S$. By the least upper bound property, $S$ has a least upper bound. Let us detone it by $c$. Since $c$ is the least upper bound for $S$ and $b$ is an upper bound, we also have that $c \leq b$. Since $a$ is in $S$ and $c$ is an upper bound, we also have that $a \leq c$. Thus $c$ is in the interval $[a,b]$. We will prove that $f(c) = 0$.

We will prove that $f(c) = 0$ by showing that $f(c)$ cannot be positive or negative.

Suppose first that $f(c)$ is positive. Since $f$ is continuous at $c$, that means that $f$ is positive on some open interval containing $c$. That means that $c$ is not the least upper bound for $S$, because any number in that interval, to the left of $c$ is also an upper bound for $S$. That is impossible and so $f(c)$ cannot be positive.




Suppose now that $f(c)$ is negative. Since $f$ is continuous at $c$, that means that $f$ is negative on some open interval containing $c$. That means that $c$ is not an upper bound for $S$, because a number in that interval, to the right of $c$ is also an element of $S$. That is impossible and so $f(c)$ cannot be negative.




So $f(c) = 0$ which completes our proof. ■

Theorem: (**The Intermediate Value Theorem for Continuous Functions**) If $f$ is continuous on a closed interval $[a,b]$ and if $y_0$ is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c$ in $[a,b]$.

Proof: Let us assume that $f(a) < f(b)$ and let $y_0$ be a fixed number between $f(a)$ and $f(b)$. We define a new function $g(x) = f(x) - y_0$. Clearly, $g$ is continuous on $[a,b]$ because it is the difference of two continuous functions. Also, $g(a)$ is negative and $g(b)$ is positive, because

$$f(a) < y_0 \qquad\qquad y_0 < f(b)$$

$$f(a) - y_0 < 0 \qquad\qquad 0 < f(b) - y_0$$

$$g(a) < 0 \qquad\qquad 0 < g(b)$$

Then, by the Intermediate Value Theorem, there exists $c$ in $(a,b)$ with $g(c) = 0$.

$$\begin{aligned} 0 &= g(c) \\ 0 &= f(c) - y_0 \\ y_0 &= f(c) \end{aligned}$$

and this completes our proof.■

Definition: Suppose that $f$ is a function and $c$ is an interior point of its domain. If the (two-sided) limit

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

exists (and is finite) we say that $f$ is **differentiable at** $c$ and denote this limit as $f'(c)$.

Theorem: If $f$ is differentiable at $a$, then it is continuous there.

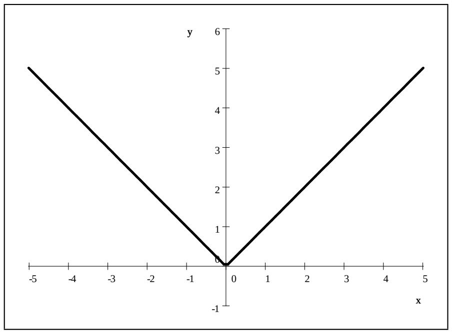
Proof: Suppose that $f$ is differentiable at a number $a$. Then $f'(a)$ exists which means that $f(a)$ exists and the limit $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ also exists and is finite. Let us start with the true statement that $0=0\cdot f'(a)$.

$$\begin{aligned}
0 &= 0\cdot f'(a) && \\
0 &= \lim_{h\to 0} h\cdot\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} && \text{by the product rule of limits}\\
0 &= \lim_{h\to 0}\left(h\cdot\frac{f(a+h)-f(a)}{h}\right) && \text{cancel out } h\\
0 &= \lim_{h\to 0}\left(f(a+h)-f(a)\right) && \text{by the difference rule of limits}\\
0 &= \lim_{h\to 0} f(a+h)-\lim_{h\to 0} f(a) && \\
\lim_{h\to 0} f(a) &= \lim_{h\to 0} f(a+h) && \text{by the constant rule of limits}\\
f(a) &= \lim_{h\to 0} f(a+h) &&
\end{aligned}$$

and $f(a)=\lim\limits_{h\to 0} f(a+h)$ means that $f$ is continuous at $a$.

So, differentiability implies continuity. What about backwards? The answer is no. Consider the function

$$f(x)=|x|=\begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } y<0\end{cases}$$




Although this function is continuous at zero, it is not differentiable there. Recall that the derivative is a two-sided limit. As $h$ approaches zero, it is negative when we compute the left-limit and positive when we compute the right limit.

$$\begin{aligned}
\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h-0}{h}=\lim_{h\to 0^-}-1=-1 \quad \text{and}\\
\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h-0}{h}=\lim_{h\to 0^+}1=1
\end{aligned}$$

So the derivative is not defined at $x=0$.

Example 1: Suppose that $f$ is a function defined as $f(x) = \begin{cases} mx - 10 & \text{if } x < -2 \\ x^2 + 9x - 8 & \text{if } x \geq -2 \end{cases}$. Find the value of $m$ if we know that $f$ is continuous everywhere.

Solution: If $x < -2$, then the function is continuous for all $x$. Similarly, $f$ is also continuous on all $x$ with $x \geq -2$. The only questionable point is at $x = -2$. For a continuous function, we need the left limit and the right limit to exist and have the same value.

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^+} f(x) \\ \lim_{x \to -2^-} (mx - 10) &= \lim_{x \to -2^+} \left(x^2 + 9x - 8\right) \end{aligned}$$

By the various properties of limits, this equation can be simplified as follows:

$$\begin{aligned} m(-2) - 10 &= (-2)^2 + 9(-2) - 8 \\ -2m - 10 &= -22 \\ -2m &= -12 \\ m &= 6 \end{aligned}$$

And so $m = 6$ is the value for which $f$ is continuous on the entire number line.

## Practice Problems

1. Suppose that $f$ is a function defined as $f(x) = \begin{cases} mx - 13 & \text{if } x < -10 \\ x^2 + 5x - 3 & \text{if } x \geq -10 \end{cases}$. Find the value of $m$ if we know that $f$ is continuous everywhere.
2. Suppose that $f$ is a function defined as $f(x) = \begin{cases} 8x - 4 & \text{if } x \leq 4 \\ -2x + b & \text{if } x > 4 \end{cases}$. Find the value of $b$ if we know that $f$ is continuous everywhere.
3. Suppose that $f$ is a function defined as $f(x) = \begin{cases} mx - 11 & \text{if } x < -6 \\ x^2 + 4x - 5 & \text{if } x \geq -6 \end{cases}$. Find the value of $m$ if we know that $f$ is continuous everywhere.
4. Suppose that $f$ is a function defined as $f(x) = \begin{cases} 2x + b & \text{if } x < 7 \\ \sqrt{x + 2} & \text{if } x \geq 7 \end{cases}$. Find the value of $b$ if we know that $f$ is continuous everywhere.

## Answers - Practice Problems

1.) $-6$ 2.) $36$ 3.) $-3$ 4) $-11$