1. $\dfrac{d}{dx}(\sin x) = \cos x$

2. $\dfrac{d}{dx}(\cos x) = -\sin x$

3. $\dfrac{d}{dx}(\tan x) = \sec^2 x = \tan^2 x + 1$

4. $\dfrac{d}{dx}(\cot x) = -\csc^2 x = -\cot^2 x - 1$

5. $\dfrac{d}{dx}(\sec x) = \dfrac{\sin x}{\cos^2 x} = \sec x \tan x$

6. $\dfrac{d}{dx}(\csc x) = -\dfrac{\cos x}{\sin^2 x} = -\csc x \cot x$

7. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$

8. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$

9. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{x^2+1}$

10. $\dfrac{d}{dx}\left(\cot^{-1} x\right) = -\dfrac{1}{x^2+1}$

11. $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$

12. $\dfrac{d}{dx}\left(\csc^{-1} x\right) = -\dfrac{1}{|x|\sqrt{x^2-1}}$

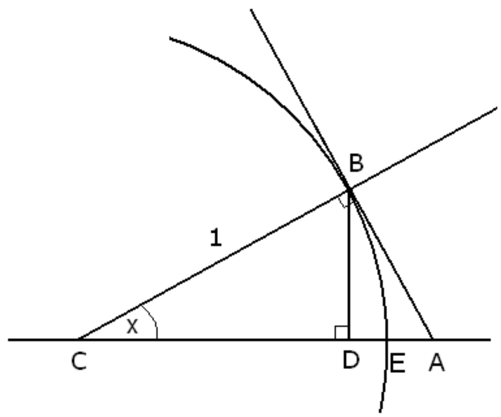
Proofs

Theorems 1 and 2: $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$

Claim 1.) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof: This theorem and the next one are necessary for differentiating $\sin x$ and $\cos x$. Recall a theorem: Let r be the radius of a circle. If α is measured in radians, then the area of a sector with a central angle of α is $A_{\text{sector}} = \frac{\alpha r^2}{2}$. (Notation: $\overline{AB}$ will denote the length of line segment AB.)

Let x be a very small positive angle, measured in radians, drawn into a unit circle as shown on the picture below. Let B be the point where the unit circle intersects the ray determined by x. We then draw a tangent line to the circle at point B. Let A be the point where the tangent line intersects the $x-$axis. We also draw a vertical line through B. Let D be the point where this vertical line intersects the $x-$axis. Finally, let us denote by E the point with coordinates $(0, 1)$.



The proof will be based on the following fact: because they include each other, the following three areas can be easily compared:

$$\text{Area of triangle } CDB \ \le \ \text{Area of sector } CEB \ \le \ \text{Area of triangle } ABC$$

Area of triangle CDB: the horizontal side, $\overline{CD} = \cos x$ and the vertical side, $\overline{DB} = \sin x$. Since this is a right triangle, the area is: $A_{CDB} = \frac{1}{2} \sin x \cos x$.

Area of sector CEB: $A_{\text{sector}} = \frac{1^2 x}{2} = \frac{x}{2}$.

Area of triangle ABC: there is a right angle at point B because the tangent line drawn to a circle is perpendicular to the radius drawn to the point of tangency. So the area is $A_{ABC} = \frac{1}{2}\overline{AB} \cdot \overline{BC}$. Clearly $\overline{BC} = 1$. To compute $\overline{AB}$, in triangle ABC, $\tan x = \frac{\overline{AB}}{1}$ and so $\overline{AB} = \tan x$.

Area of triangle ABC: $\frac{1}{2}(1)(\tan x) = \frac{\tan x}{2}$ or $\frac{\sin x}{2 \cos x}$. So now

$$\text{Area of triangle } CDB \ \le \ \text{Area of sector } CEB \ \le \ \text{Area of triangle } ABC$$

translates to

$$\frac{1}{2} \sin x \cos x \le \frac{x}{2} \le \frac{\sin x}{2 \cos x}$$

Let us divide all three sides by $\dfrac{\sin x}{2}$. Because x is small and positive, $\dfrac{\sin x}{2}$ is positive and so we do not need to reverse the inequality signs.

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

Suppose now that x approaches zero. Then both $\cos x$ and $\dfrac{1}{\cos x}$ approach 1. By the sandwich principle, $\dfrac{x}{\sin x}$, the quantity locked in between those two must also approach 1.

$$\begin{array}{ccccc} \cos x & \le & \dfrac{x}{\sin x} & \le & \dfrac{1}{\cos x} \\ \downarrow & & & & \downarrow \\ 1 & & & & 1 \end{array}$$

If $\dfrac{x}{\sin x}$ approaches 1, so is its reciprocal, $\dfrac{\sin x}{x}$.

So far, we have proven the statement for positive values of x, that is, $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{x} = 1$. A similar argument works for negative values of x.

Claim 2.) $\displaystyle\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$

Proof:

$$\begin{aligned} \lim_{x\to 0}\frac{\cos x - 1}{x} &= \lim_{x\to 0}\frac{\cos x - 1}{x}\cdot 1 = \lim_{x\to 0}\left(\frac{\cos x - 1}{x}\cdot\frac{\cos x + 1}{\cos x + 1}\right) = \lim_{x\to 0}\frac{\cos^2 x - 1}{x\left(\cos x + 1\right)} = \lim_{x\to 0}\frac{-\left(1-\cos^2 x\right)}{x\left(\cos x + 1\right)} \\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x\left(\cos x + 1\right)} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{-\sin x}{\cos x + 1} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{-\sin x}{\cos x + 1} = 1\cdot 0 = 0 \end{aligned}$$

We are now ready to prove that $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$

Proof:

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \\ &= \lim_{h\to 0}\left(\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right) = \lim_{h\to 0}\frac{\sin x\left(\cos h - 1\right)}{h} + \lim_{h\to 0}\cos x\frac{\sin h}{h} \\ &= \sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h} = \sin x\cdot 0 + \cos x\cdot 1 = \cos x \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\ &= \lim_{h\to 0}\left(\frac{\cos x\cos h - \cos x}{h} - \frac{\sin x\sin h}{h}\right) = \lim_{h\to 0}\frac{\cos x\left(\cos h - 1\right)}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h} \\ &= \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} = \cos x\cdot 0 - \sin x\cdot 1 = -\sin x \end{aligned}$$

Theorem 3 and 4: $\dfrac{d}{dx}(\tan x) = \sec^2 x = \tan^2 x + 1$ and $\dfrac{d}{dx}(\cot x) = -\csc^2 x = -\cot^2 x - 1$.

Proof: We write $\tan x = \dfrac{\sin x}{\cos x}$ and apply the quotient rule.

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left(\frac{d}{dx}\sin x\right)\cos x - \left(\frac{d}{dx}\cos x\right)\sin x}{\cos^2 x} = \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

We will now prove $\dfrac{1}{\cos^2 x} = \tan^2 x + 1$, which is a very important connection. Looking at the previous computation,

$$\frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

The proof for $\dfrac{d}{dx}(\cot x) = -\cot^2 x - 1 = -\csc^2 x$ is very similar. We apply the quotient rule.

$$\begin{aligned}\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\left(\frac{d}{dx}\cos x\right)\sin x - \cos x\left(\frac{d}{dx}\sin x\right)}{\sin^2 x} = \frac{-\sin x \sin x - \cos x(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} = -\csc^2 x\end{aligned}$$

Also,

$$\frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-\sin^2 x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = -1 - \cot^2 x$$

Theorems 5 and 6: $\dfrac{d}{dx}(\sec x) = \sec x \tan x$ and $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

Proof: We write $\sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$ and apply the chain rule.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left((\cos x)^{-1}\right) = -1(\cos x)^{-2}\left(\frac{d}{dx}(\cos x)\right) = \frac{-1}{\cos^2 x}(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x$$

The proof for $\dfrac{d}{dx}\csc x$ is virtually identical: we apply the chain rule.

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left((\sin x)^{-1}\right) = -1(\sin x)^{-2}\left(\frac{d}{dx}\sin x\right) = \frac{-1}{\sin^2 x}\cos x = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\cot x \csc x$$

Note: why do we prefer the form $\sec x \tan x$ over the form $\dfrac{\sin x}{\cos^2 x}$? One of the reasons is the adventage we'll see in differentiating the inverse functions $\sec^{-1} x$ and $\csc^{-1} x$.

Theorems 7 and 8: $\dfrac{d}{dx}\left(\sin^{-1}x\right)=\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{d}{dx}\left(\cos^{-1}x\right)=-\dfrac{1}{\sqrt{1-x^2}}$

Proof: Recall that when we compose a function f with its inverse f^{-1}, the result is always the same function, (also called the identity function, $id\,(x)=x$)

$$f\left(f^{-1}(x)\right)=x$$

We will state this fact for $f(x)=\sin x$ and differentiate both sides of the equation. For the left-hand side, we use the chain rule.

$$\begin{aligned}\sin\left(\sin^{-1}x\right) &= x\\ \cos\left(\sin^{-1}x\right)\cdot\frac{d}{dx}\sin^{-1}x &= 1 \qquad\qquad \text{divide by } \cos\left(\sin^{-1}x\right)\\ \frac{d}{dx}\sin^{-1}x &= \frac{1}{\cos\left(\sin^{-1}x\right)}\end{aligned}$$

We now need to simplify $\cos\left(\sin^{-1}x\right)$. We will present two methods to simplify this expression.

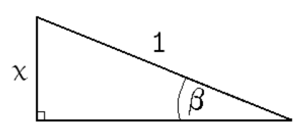
Method 1. We first introduce a new variable, β. Let $\beta=\sin^{-1}x$. This means that $-\dfrac{\pi}{2}<\beta<\dfrac{\pi}{2}$, and $\sin\beta=x$. We need to simplify $\cos\underbrace{\left(\sin^{-1}x\right)}_{\beta}=\cos\beta$. Since $\sin^2\beta+\cos^2\beta=1$,

$$\cos\beta=\pm\sqrt{1-\sin^2\beta}=\pm\sqrt{1-x^2}$$

Since $-\dfrac{\pi}{2}<\beta<\dfrac{\pi}{2}$, $\cos\beta$ is positive and so $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$.

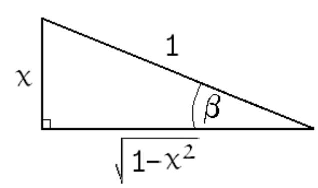
Method 2. We first introduce a new variable, β. Let $\beta=\sin^{-1}x$. This means that $-\dfrac{\pi}{2}<\beta<\dfrac{\pi}{2}$, and $\sin\beta=x$. Now the goal is to simplify $\cos\underbrace{\left(\sin^{-1}x\right)}_{\beta}=\cos\beta$.

We will use a right triangle to find the expression - up to its sign. We first draw a right triangle in which $\sin\beta=x=\dfrac{x}{1}$.



Next we use the Pythagoream Theorem to find the missing side to be $\sqrt{1-x^2}$.

The adventage of this method is that now we can read *any* trigonometric function value of $\beta=\sin^{-1}x$ using this right triangle.



From the triangle,

$$\cos\beta=\cos\left(\sin^{-1}x\right)=\frac{\sqrt{1-x^2}}{1}=\sqrt{1-x^2}$$

The answer at this point is really $\pm\sqrt{1-x^2}$ as the triangle gave us the answer only up to a sign. For the sign, we need to argue using the location of β on the unit circle. Since $\beta=\sin^{-1}x$, $-\dfrac{\pi}{2}\le\beta\le\dfrac{\pi}{2}$. Thus β is in the first or in the fourth quadrant. In both cases, cosine is positive, thus $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$.

Thus $\dfrac{d}{dx}\left(\sin^{-1}x\right)=\dfrac{1}{\cos\left(\sin^{-1}x\right)}=\dfrac{1}{\sqrt{1-x^2}}$.

The proof for $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$ is virtually identical.

Proof: Recall that when we compose a function f with its inverse f^{-1}, the result is always the same function.

$$f\left(f^{-1}(x)\right) = x$$

We will state this fact for $f(x) = \cos x$ and differentiate both sides of the equation. For the left-hand side, we use the chain rule.

$$\begin{aligned} \cos\left(\cos^{-1} x\right) &= x \\ -\sin\left(\cos^{-1} x\right) \cdot \frac{d}{dx}\left(\cos^{-1} x\right) &= 1 \qquad \text{divide by } \sin\left(\cos^{-1} x\right) \\ \frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sin\left(\cos^{-1} x\right)} \end{aligned}$$

We now need to simplify the expression $\sin\left(\cos^{-1} x\right)$. We will present two methods for this.

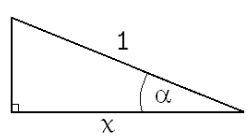
Method 1. Let $\alpha = \cos^{-1} x$. Then $x = \cos \alpha$ and α is between 0 and π.

$$\sin\left(\underbrace{\cos^{-1} x}_{\alpha}\right) = \sin \alpha = \pm\sqrt{1-\cos^2 \alpha} = \pm\sqrt{1-x^2}$$

Since α is between 0 and π, $\sin \alpha$ is positive and so $\sin \alpha = \sqrt{1-x^2}$.

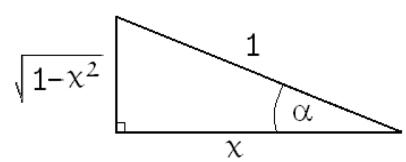
Method 2. Let $\alpha = \cos^{-1} x$. Then $x = \cos \alpha$ and α is between 0 and π.

We first draw a triangle in which $\cos \alpha = x = \dfrac{x}{1}$. Please note that every time we approach such a trigonometric question using a right triangle, our answer would be accurate up to sign - for the sign we would have to argue separately.



We find the missing side via the Pythagoream Theorem: $\sqrt{1-x^2}$.

Now we can read *any* trigonometric function value using this triangle.

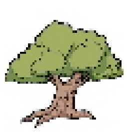


Now we read sine from the triangle:

$$\sin \alpha = \sin\left(\cos^{-1} x\right) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

The answer at this point is really $\pm\sqrt{1-x^2}$ as the triangle gave us the answer only up to a sign. For the sign, we need to argue using the location of α on the unit circle. Since $\alpha = \cos^{-1} x$, $0 \leq \alpha \leq \pi$. Thus α is in the first or in the second quadrant. In both cases, sine is positive, thus $\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}$.

Consequently, $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sin\left(\cos^{-1} x\right)} = -\dfrac{1}{\sqrt{1-x^2}}$



Enrichment

If $\dfrac{d}{dx}\left(\sin^{-1} x\right)$ and $\dfrac{d}{dx}\left(\cos^{-1} x\right)$ are opposites, then what can be said about the function $f(x) = \sin^{-1} x + \cos^{-1} x$?

Theorems 9 and 10: $\dfrac{d}{dx}\left(\tan^{-1}x\right)=\dfrac{1}{x^2+1}$ and $\dfrac{d}{dx}\left(\cot^{-1}x\right)=-\dfrac{1}{x^2+1}$

Proof: Recall that $\dfrac{d}{dx}(\tan x)=\sec^2 x=\tan^2 x+1$. Also recall that when we compose a function f with its inverse $f^{-1},$ the result is always the same function.

$$f\left(f^{-1}(x)\right)=x$$

We will state this fact for $f(x)=\tan x$ and differentiate both sides of the equation. For the left-hand side, we use the chain rule.

$$\begin{aligned}
\tan\left(\tan^{-1}x\right) &= x \\
\sec^2\left(\tan^{-1}x\right)\cdot\frac{d}{dx}\left(\tan^{-1}x\right) &= 1 \\
\left(\tan^2\left(\tan^{-1}x\right)+1\right)\cdot\frac{d}{dx}\left(\tan^{-1}x\right) &= 1 && \tan\left(\tan^{-1}x\right)=x \\
\left(x^2+1\right)\cdot\frac{d}{dx}\left(\tan^{-1}x\right) &= 1 && \text{divide by } x^2+1 \\
\frac{d}{dx}\left(\tan^{-1}x\right) &= \frac{1}{x^2+1}
\end{aligned}$$

The proof for $\dfrac{d}{dx}\left(\cot^{-1}x\right)=-\dfrac{1}{x^2+1}$ is virtually identical. We compose the function $\cot x$ with its inverse $\cot^{-1}x$ and differentiate. Recall that $\dfrac{d}{dx}\cot x=-\cot^2 x-1$

$$\begin{aligned}
\cot\left(\cot^{-1}x\right) &= x \\
\left(-\cot^2\left(\cot^{-1}x\right)-1\right)\cdot\frac{d}{dx}\left(\cot^{-1}x\right) &= 1 && \cot\left(\cot^{-1}x\right)=x \\
\left(-x^2-1\right)\cdot\frac{d}{dx}\left(\cot^{-1}x\right) &= 1 && \text{divide by } -x^2-1 \\
\frac{d}{dx}\left(\cot^{-1}x\right) &= \frac{1}{-x^2-1}=-\frac{1}{x^2+1}
\end{aligned}$$

Theorem 11 and 12: $\dfrac{d}{dx}\left(\sec^{-1}x\right)=\dfrac{1}{|x|\sqrt{x^2-1}}$ and $\dfrac{d}{dx}\left(\csc^{-1}x\right)=-\dfrac{1}{|x|\sqrt{x^2-1}}$

Proof: We compose the function $\sec x$ with its inverse $\sec^{-1}x$ and differentiate. Recall that $\dfrac{d}{dx}(\sec x)=\sec x\tan x$.

$$\begin{aligned}
\sec\left(\sec^{-1}x\right) &= x \\
\sec\left(\sec^{-1}x\right)\tan\left(\sec^{-1}x\right)\cdot\frac{d}{dx}\left(\sec^{-1}x\right) &= 1 && \sec\left(\sec^{-1}x\right)=x \\
x\tan\left(\sec^{-1}x\right)\cdot\frac{d}{dx}\left(\sec^{-1}x\right) &= 1 && \text{divide by } x\tan\left(\sec^{-1}x\right) \\
\frac{d}{dx}\left(\sec^{-1}x\right) &= \frac{1}{x\tan\left(\sec^{-1}x\right)}
\end{aligned}$$

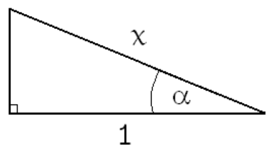
We now just need to simplify the expression $\tan\left(\sec^{-1}x\right)$.

Method 1. To simplify $\tan\left(\sec^{-1}x\right)$, we introduce a new variable α. Let $\alpha = \sec^{-1}x$. Then we have $\tan\Big(\underbrace{\sec^{-1}x}_{\alpha}\Big) = \tan\alpha$ where $\sec\alpha = x$ and α is between 0 and π. Recall that $\sec^2\alpha = \tan^2\alpha + 1$. If we don't have this formula memorized, we can easily derive it from the Pythagorean identity.

$$\begin{aligned} \sin^2\alpha + \cos^2\alpha &= 1 && \text{divide by } \cos^2 x \\ \frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} &= \frac{1}{\cos^2\alpha} \\ \tan^2\alpha + 1 &= \sec^2\alpha \\ \tan\alpha &= \pm\sqrt{\sec^2\alpha - 1} \end{aligned}$$

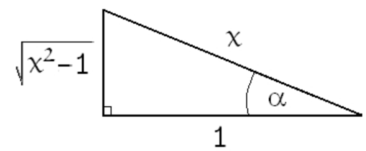
Method 2. To simplify $\tan\left(\sec^{-1}x\right)$, we introduce a new variable α. Let $\alpha = \sec^{-1}x$. Then $\sec\alpha = x$ and α is between 0 and π. Then we need to compute $\tan\alpha$.

We draw a right triangle in which $\sec\alpha = x = \dfrac{x}{1}$.



We find the missing side via the Pythagoream Theorem: $\sqrt{x^2-1}$.

Now we can read *any* trigonometric function value using this triangle.



Now we read from the triangle:

$$\tan\alpha = \tan\left(\sec^{-1}x\right) = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

The answer at this point is really $\pm\sqrt{x^2-1}$ as the triangle gave us the answer only up to a sign. Thus the derivative is

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\left(\pm\sqrt{x^2-1}\right)} = \pm\frac{1}{x\sqrt{x^2-1}}$$

We now need to figure out the sign of the derivative. From the graph of $\sec^{-1}x$ we can see that it is strictly increasing on both intervals making up its domain, thus the derivative is always positive. If x is positive, then $\dfrac{d}{dx}\left(\sec^{-1}x\right) = \dfrac{1}{x\sqrt{x^2-1}}$ and if x is negative, then $\dfrac{d}{dx}\left(\sec^{-1}x\right) = -\dfrac{1}{x\sqrt{x^2-1}}$. This can be expressed in a shorter form as

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

The proof for $\dfrac{d}{dx}\left(\csc^{-1}x\right) = -\dfrac{1}{|x|\sqrt{x^2-1}}$ is virtually identical. As before, we compose the function $\csc x$ with its inverse and differentiate. Recall that $\dfrac{d}{dx}(\csc x) = -\csc x\cot x$

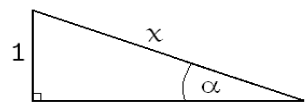
$$\begin{aligned} \csc\left(\csc^{-1}x\right) &= x \\ -\csc\left(\csc^{-1}x\right)\cot\left(\csc^{-1}x\right)\cdot\frac{d}{dx}\left(\csc^{-1}x\right) &= 1 && \csc\left(\csc^{-1}x\right) = x \\ -x\cot\left(\csc^{-1}x\right)\cdot\frac{d}{dx}\left(\csc^{-1}x\right) &= 1 && \text{divide by } -x\cot\left(\csc^{-1}x\right) \\ \frac{d}{dx}\left(\csc^{-1}x\right) &= -\frac{1}{x\cot\left(\csc^{-1}x\right)} \end{aligned}$$

We need to simplify $\cot\left(\csc^{-1}x\right)$. Let $\alpha=\csc^{-1}x$. $\cot\left(\underbrace{\csc^{-1}x}_{\alpha}\right)=\cot\alpha$ where $\csc\alpha=x$ and α is between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

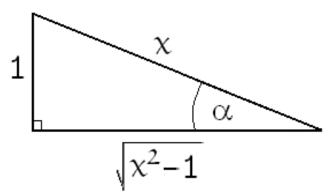
Method 1. We start with the Pythagorean identity and divide both sides by $\sin^2\alpha$.

$$\begin{aligned}
\sin^2\alpha+\cos^2\alpha &= 1 \qquad \text{divide by } \sin^2 x\\
\frac{\sin^2\alpha}{\sin^2\alpha}+\frac{\cos^2\alpha}{\sin^2\alpha} &= \frac{1}{\sin^2\alpha}\\
1+\cot^2\alpha &= \csc^2\alpha\\
\cot^2\alpha &= \csc^2\alpha-1\\
\cot\alpha &= \pm\sqrt{\csc^2\alpha-1}
\end{aligned}$$

Method 2. We draw a right triangle in which $\csc\alpha=x=\dfrac{x}{1}$.



We find the missing side using the Pythagorean Theorem and read the desired trigonometric function value.



Thus the derivative is

$$\frac{d}{dx}\left(\csc^{-1}x\right)=-\frac{1}{x\left(\pm\sqrt{x^2-1}\right)}=\pm\frac{1}{x\sqrt{x^2-1}}$$

From the graph of $\csc^{-1}x$ we can see that it is strictly decreasing on both intervals of its domain, thus the derivative is always negative. If x is positive, then $\dfrac{d}{dx}\left(\csc^{-1}x\right)=-\dfrac{1}{x\sqrt{x^2-1}}$ and if x is negative, then $\dfrac{d}{dx}\left(\csc^{-1}x\right)=\dfrac{1}{x\sqrt{x^2-1}}$. This can be expressed in a shorter form as

$$\frac{d}{dx}\left(\csc^{-1}x\right)=-\frac{1}{|x|\sqrt{x^2-1}}$$

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