# Sample Problems

1. A city is of a circular shape. The area of the city is growing at a constant rate of $2\dfrac{\text{mi}^2}{\text{y}}$ (square miles per year). How fast is the radius growing when it is exactly $15\,\text{mi}$?

2. A sphere is growing in such a manner that its radius increases at $0.2\dfrac{\text{m}}{\text{s}}$ (meters per second). How fast is its volume increasing when its radius is $4\,\text{m}$ long?

3. A sphere is growing in such a manner that its volume increases at $0.2\dfrac{\text{m}^3}{\text{s}}$ (cubic meters per second). How fast is its radius increasing when it is $7\,\text{m}$ long?

4. A cube is decreasing in size so that its surface is changing at a constant rate of $-0.5\dfrac{\text{m}^2}{\text{min}}$. How fast is the volume of the cube changing when it is $27\,\text{m}^3$?

5. A ladder $20\,\text{ft}$ long leans against a vertical building. If the top of the ladder slides down at a rate of $\sqrt{3}\dfrac{\text{ft}}{\text{s}}$, how fast is the bottom of the ladder sliding away from the building when the top of the ladder is $10\,\text{ft}$ above the ground?

6. A tank, shaped like a cone shown on the picture, is being filled up with water. The top of the tank is a circle with radius $5\,\text{ft}$, its height is $15\,\text{ft}$. Water is added to the tank at the rate of $V'(t) = 2\pi\dfrac{\text{ft}^3}{\text{min}}$. How fast is the water level rising when the water level is 6 ft high? (The volume of a cone with height h and base radius r is $V = \dfrac{\pi r^2 h}{3}$.)
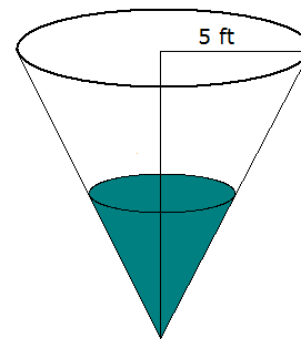
7. A rotating light is located 18 feet from a wall. The light completes one rotation every 5 seconds. Find the rate at which the light projected onto the wall is moving along the wall when the light's angle is 5 degrees from perpendicular to the wall.
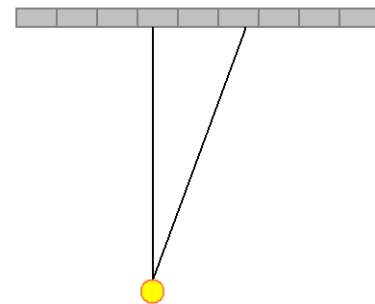
8. The altitude of a triangle is increasing at a rate of 2.2 centimeters/minute while the area of the triangle is increasing at a rate of 1.5 square centimeters/minute. At what rate is the base of the triangle changing when the altitude is 11 centimeters and the area is 87 square centimeters?

9. The area of a rectangle is kept fixed at 100 square meters while the legths of the sides vary. Express the rate of change of the length of the vertical side in terms of the rate of change in the length of the other side when

    a) the horizontal side is 18 meters long   b) the rectangle is a square

10. Two quantities p and q depending on t are subject to the relation $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

    a) Express $p'(t)$ in terms of $q'(t)$.

    b) At a certain moment, $p(t_0) = \dfrac{4}{3}$ and $p'(t_0) = 2$. Find $q(t_0)$ and $q'(t_0)$.

11. The base radius and height of a cylinder are constantly changing but the volume of the cylinder is kept at a constant 600π in^3.

    a) At a time t_1 the base radius is $r(t_1) = 10$ in and its rate of change is $r'(t_1) = 0.2\dfrac{\text{in}}{\text{s}}$. Compute the rate of change of the height of the cylinder $h(t)$ at time t_1.

    b) At a time t_2 the height is $h(t_2) = 12$ in and its rate of change is $r'(t_2) = -0.5\dfrac{\text{in}}{\text{s}}$. Compute the rate of change of the radius of the cylinder $r(t)$ at time t_2.

12. An object, dropped from a height of h has a location of $y(t) = -16t^2 + h$ feet after t seconds. We dropped a small object from a height of 60 feet.

    a) Where is the object and what is its velocity after 1.5 seconds?

    b) Suppose there is a 30 feet tall street light 10 feet away from the point where the object will land. How far is the shadow of the object from the base of the street light at $t = 1.5$?

    c) How fast is the obejct's shadow moving at $t = 1.5$?

30 ft

10 ft
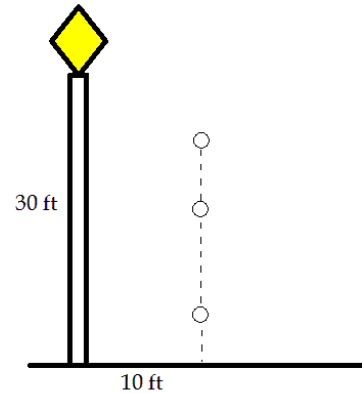
Sample Problems - Answers

1.) $\dfrac{1}{15\pi}\dfrac{\text{mi}}{\text{y}}$ 2.) $12.8\pi\dfrac{\text{m}^3}{\text{s}} \approx 40.2123859659494\dfrac{\text{m}^3}{\text{s}}$ 3.) $\dfrac{1}{980\pi}\dfrac{\text{m}}{\text{s}} \approx 3.24806 \times 10^{-4}\dfrac{\text{m}}{\text{s}}$

4.) $-0.375\dfrac{\text{m}^3}{\text{min}}$ 5.) $1\dfrac{\text{ft}}{\text{s}}$ 6.) $\dfrac{1}{2}\dfrac{\text{ft}}{\text{min}}$ 7.) $\dfrac{36\pi}{5}\sec^2(5°)\dfrac{\text{ft}}{\text{s}} \approx 22.7926\dfrac{\text{ft}}{\text{s}}$ 8.) $-2.89091\dfrac{\text{cm}}{\text{min}}$

9.) a) $v'(t) = -\dfrac{25}{81}h'(t)$ b) $v'(t) = -h'(t)$ 10.) a) $p' = -\dfrac{p^2}{q^2}q'$ b) -18

11.) a) $-0.24\dfrac{\text{in}}{\text{s}}$ b) $\dfrac{5}{48}\sqrt{2}\dfrac{\text{in}}{\text{s}} \approx 0.147314\dfrac{\text{in}}{\text{s}}$

12.) a) $y(1.5) = 24\,\text{ft}$ and $y'(1.5) = -48\dfrac{\text{ft}}{\text{s}}$ b) $50\,\text{ft}$ c) $-400\dfrac{\text{ft}}{\text{s}}$

Sample Problems - Solutions

1. A city is of a circular shape. The area of the city is growing at a constant rate of $2\dfrac{\text{mi}^2}{\text{y}}$ (square miles per year). How fast is the radius growing when it is exactly $15\,\text{mi}$?

   Solution: The area of a circle with radius r is $A = \pi r^2$. Only this time, both A and r are functions of time: $A(t) = \pi r^2(t)$. It is also given that $A'(t) = 2\dfrac{\text{mi}^2}{\text{y}}$. We differentiate both sides of $A(t) = \pi r^2(t)$ with respect to t.

$$A(t) = \pi r^2(t)$$
$$A'(t) = 2\pi r(t)\, r'(t)$$

Let t_1 be the time when $r(t_1) = 15\,\text{mi}$. Then

$$A'(t_1) = 2\pi r(t_1)\, r'(t_1)$$

$$\dfrac{A'(t_1)}{2\pi r(t_1)} = r'(t_1) \implies r'(t_1) = \dfrac{A'(t_1)}{2\pi r(t_1)} = \dfrac{2\dfrac{\text{mi}^2}{\text{y}}}{2\pi 15\,\text{mi}} = \dfrac{1}{15\pi}\dfrac{\text{mi}}{\text{y}}$$

2. A sphere is growing in such a manner that its radius increases at $0.2\dfrac{\text{m}}{\text{s}}$ (meters per second). How fast is its volume increasing when its radius is $4\,\text{m}$ long?

$$V(t) = \dfrac{4}{3}\pi r^3(t)$$
$$V'(t) = \dfrac{4}{3}\pi\left(3r^2(t)\, r'(t)\right) = 4\pi r^2(t)\, r'(t)$$

Let t_1 be the time when $r(t_1) = 4\,\text{m}$. Then

$$V'(t_1) = 4\pi r^2(t_1)\, r'(t_1) = 4\pi(4\,\text{m})^2\left(0.2\dfrac{\text{m}}{\text{s}}\right) = 12.8\pi\dfrac{\text{m}^3}{\text{s}} \approx 40.212386\dfrac{\text{m}^3}{\text{s}}$$

3. A sphere is growing in such a manner that its volume increases at $0.2 \frac{m^3}{s}$ (cubic meters per second). How fast is its radius increasing when it is $7\,m$ long?

$$
\begin{aligned}
V\left(t\right) &= \frac{4}{3}\pi r^3\left(t\right) & \text{or} & & V &= \frac{4}{3}\pi r^3 \\
V'\left(t\right) &= \frac{4}{3}\pi\left(3r^2\left(t\right)r'\left(t\right)\right) & & & V' &= \frac{4}{3}\pi\left(3r^2 r'\right) \\
V'\left(t\right) &= 4\pi r^2\left(t\right)r'\left(t\right) & & & V' &= 4\pi r^2 r' \\
r'\left(t\right) &= \frac{V'\left(t\right)}{4\pi r^2\left(t\right)} & & & r' &= \frac{V'}{4\pi r^2}
\end{aligned}
$$

Let t_1 be the time when $r\left(t_1\right) = 7\,m$. Then

$$
r'\left(t_1\right) = \frac{V'\left(t_1\right)}{4\pi r^2\left(t_1\right)} = \frac{0.2\frac{m^3}{s}}{4\pi\left(7\,m\right)^2} \approx 0.00032481\frac{m}{s}
$$

4. A cube is decreasing in size so that its surface is changing at a constant rate of $-0.5\frac{m^2}{min}$. How fast is the volume of the cube changing when it is $27\,m^3$?

Solution: Let $s\left(t\right)$ denote the length of the edges of the cube at a time t. Let $A\left(t\right)$ denote the surface area of the cube at a time t, and $V\left(t\right)$ denote the volume of the cube. Recall the formulas $A = 6s^2$ and $V = s^3$.

$$
\begin{aligned}
A\left(t\right) &= 6s^2\left(t\right) \\
A'\left(t\right) &= 6\cdot 2s\left(t\right)s'\left(t\right) \\
\frac{A'\left(t\right)}{12s\left(t\right)} &= s'\left(t\right)
\end{aligned}
$$

$$
\begin{aligned}
V\left(t\right) &= s^3\left(t\right) \\
V'\left(t\right) &= 3s^2\left(t\right)s'\left(t\right) = 3s^2\left(t\right)\cdot\frac{A'\left(t\right)}{12s\left(t\right)} = \frac{s\left(t\right)A'\left(t\right)}{4}
\end{aligned}
$$

Let t_0 be the time at which the volume is $27\,m^3$. At that time, the edges are $3\,m$ long.

$$
V'\left(t_0\right) = \frac{s\left(t_0\right)A'\left(t_0\right)}{4} = \frac{3\,m\cdot\left(-0.5\frac{m^2}{min}\right)}{4} = -0.375\frac{m^3}{min}
$$

5. A ladder $20\,ft$ long leans against a vertical building. If the top of the ladder slides down at a rate of $\sqrt{3}\frac{ft}{s}$, how fast is the bottom of the ladder sliding away from the building when the top of the ladder is $10\,ft$ above the ground?

Solution: Let $x\left(t\right)$ and $y\left(t\right)$ denote the horizontal and vertical position of the endpoints of the ladder. With this notation, $y'\left(t\right) = -\sqrt{3}\frac{ft}{s}$. By the Pythagorean theorem, $x\left(t\right)^2 + y\left(t\right)^2 = 400$. We differentiate both sides and solve for $x'\left(t\right)$.

$$
\begin{aligned}
x\left(t\right)^2 + y\left(t\right)^2 &= 400 & \text{or} & & x^2 + y^2 &= 400 \\
2x\left(t\right)x'\left(t\right) + 2y\left(t\right)y'\left(t\right) &= 0 & & & 2xx' + 2yy' &= 0 \\
x'\left(t\right) &= -\frac{y\left(t\right)y'\left(t\right)}{x\left(t\right)} & & & x' &= -\frac{yy'}{x}
\end{aligned}
$$

Let t_1 denote the time when $x(t_1) = 10\,\text{ft}$.

$$x'(t_1) = -\frac{y(t_1)\,y'(t_1)}{x(t_1)} = -\frac{(10\,\text{ft})\left(-\sqrt{3}\,\frac{\text{ft}}{\text{s}}\right)}{\sqrt{(20\,\text{ft})^2 - (10\,\text{ft})^2}} = -\frac{-10\sqrt{3}\,\frac{\text{ft}^2}{\text{s}}}{\sqrt{300\,\text{ft}^2}} = 1\frac{\text{ft}}{\text{s}}$$

6. A tank, shaped like a cone shown on the picture, is being filled up with water. The top of the tank is a circle with radius $5\,\text{ft}$, its height is $15\,\text{ft}$. Water is added to the tank at the rate of $V'(t) = 2\pi\,\dfrac{\text{ft}^3}{\text{min}}$. How fast is the water level rising when the water level is 6 ft high? (The volume of a cone with height h and base radius r is $V = \dfrac{\pi r^2 h}{3}$.)

5 ft
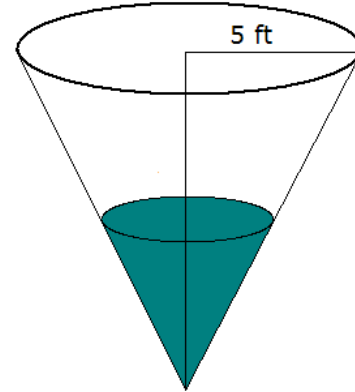
Solution: Let $h(t)$ and $r(t)$ denote the height and radius of the surface. There is a simple connection between these two:
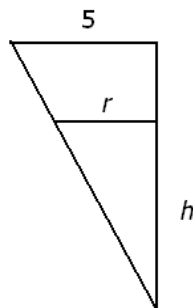
The two triangles shown on the picture above are similar. $\dfrac{r}{5} = \dfrac{h}{15}$ and so $r = \dfrac{h}{3}$.

$$V(t) = \frac{1}{3}\pi r^2(t)\,h(t) = \frac{1}{3}\pi \left(\frac{h(t)}{3}\right)^2 h(t) = \frac{1}{27}\pi h^3(t)$$

$$V'(t) = \frac{1}{27}\pi \left(3h^2(t)\,h'(t)\right) = \frac{1}{9}\pi h^2(t)\,h'(t)$$

$$V'(t) = \frac{1}{9}\pi h^2(t)\,h'(t)$$

$$\frac{9V'(t)}{\pi h^2(t)} = h'(t)$$

Let t_1 denote the time when $h(t_1) = 6\,\text{ft}$.

$$h'(t_1) = \frac{9V'(t_1)}{\pi h^2(t_1)} = \frac{9\left(2\pi\,\dfrac{\text{ft}^3}{\text{min}}\right)}{\pi\,(6\,\text{ft})^2} = \frac{1}{2}\,\frac{\text{ft}}{\text{min}}$$

7. A rotating light is located 18 feet from a wall. The light completes one rotation every 5 seconds. Find the rate at which the light projected onto the wall is moving along the wall when the light's angle is 5 degrees from perpendicular to the wall.

Solution: Let x denote the distance one the wall between the location of the light and the perpendicular distance. Let θ denote the angle from the perpendicular. Using this notation, we are given $\dfrac{d\theta}{dt}$ and we are asked $\dfrac{dx}{dt}$. We need to find how the two quantities, $\theta(t)$ and $x(t)$ are related to each other. It is given that $\theta'(t) = \dfrac{2\pi \text{ rad}}{5 \text{ s}}$ for all t.
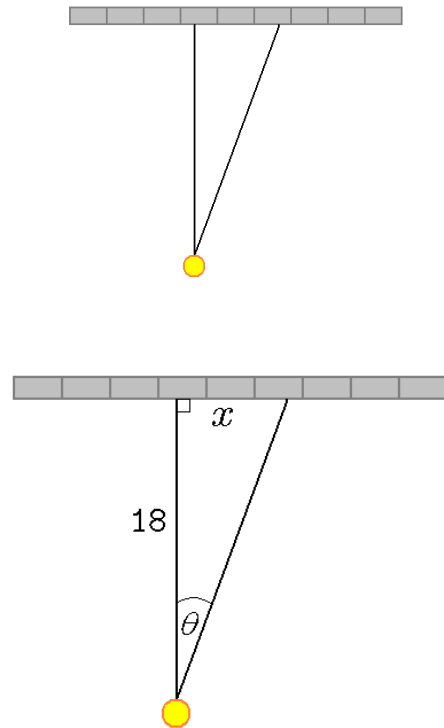
$$\tan(\theta(t)) = \frac{x(t)}{18\,\text{ft}}$$

We differentiate both sides with respect to t. In the left-hand side, we apply the chain rule:

$$\sec^2(\theta(t)) \cdot \theta'(t) = \frac{x'(t)}{18\,\text{ft}}$$
$$(18\,\text{ft})\sec^2(\theta(t)) \cdot \theta'(t) = x'(t)$$

Let t_0 be the time when $\theta(t_0) = 5°$. Then

$$x'(t_0) = (18\,\text{ft})\sec^2(\theta(t_0)) \cdot \theta'(t_0) = (18\,\text{m})\sec^2(5°) \cdot \frac{2\pi \text{ rad}}{5 \text{ s}}$$

The exact value of the answer is $\dfrac{36\pi}{5}\sec^2(5°)\,\dfrac{\text{ft}}{\text{s}}$. The approximate value is $22.7926\,\dfrac{\text{ft}}{\text{s}}$

8. The altitude of a triangle is increasing at a rate of 2.2 centimeters/minute while the area of the triangle is increasing at a rate of 1.5 square centimeters/minute. At what rate is the base of the triangle changing when the altitude is 11 centimeters and the area is 87 square centimeters?

Solution: The formula for the area of a triangle is $A = \dfrac{1}{2}bh$. In this case, these quantities are functions of time, i.e. they are $A(t)$, $b(t)$, and $h(t)$. We solve for b.

$$
\begin{aligned}
A &= \frac{1}{2}bh \\
2A &= bh \\
b &= \frac{2A}{h} \text{ or rather } b(t) = \frac{2A(t)}{h(t)}
\end{aligned}
$$

We differentiate both sides. We will use the quotient rule.

$$b'(t) = 2 \cdot \frac{A'(t)\,h(t) - A(t)\,h'(t)}{h^2(t)} = 2 \cdot \frac{\left(1.5\frac{\text{cm}^2}{\text{min}}\right)(11\,\text{cm}) - (87\,\text{cm}^2)\left(2.2\frac{\text{cm}}{\text{min}}\right)}{(11\,\text{cm})^2} \approx -2.89091\frac{\text{cm}}{\text{min}}$$

The negative sign indicates that at the time indicated, the side b is becoming shorter at a rate of 2.89091 centimeters per minute.

9. The area of a rectangle is kept fixed at 100 square meters while the legths of the sides vary. Express the rate of change of the length of the vertical side in terms of the rate of change in the length of the other side when

   a) the horizontal side is 18 meters long   b) the rectangle is a square

   Solution: Let v and h denote the length of the horizontal and vertical sides, respectively. Then clearly $vh = 100$. We differentiate both sides of this with respect of time, and then solve for v' in terms of h'.

$$\begin{aligned} h(t)\,v(t) &= 100 & \text{or} & & vh &= 100 \\ h'(t)\,v(t) + h(t)\,v'(t) &= 0 & & & v'h + vh' &= 0 \\ v'(t) &= -\frac{h'(t)\,v(t)}{h(t)} & & & v' &= -\frac{h'v}{h} \end{aligned}$$

   a) When $h = 18$, then $v = \dfrac{100}{18} = \dfrac{50}{9}$ and so

$$v'(t) = -\frac{h'(t)\,v(t)}{h(t)} = -\frac{h'(t)\left(\dfrac{50}{9}\right)}{18} = -\frac{25}{81}h'(t)$$

   b) When the rectangle is a square, then $v = h = 10$.

$$v'(t) = -\frac{h'(t)\,v(t)}{h(t)} = -\frac{h'(t)\,(10)}{10} = -h'(t)$$

10. Two quantities p and q depending on t are subject to the relation $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

    a) Express p' in terms of q'.

$$\frac{1}{p} + \frac{1}{q} = 0 \quad \Longrightarrow \quad -\frac{p'}{p^2} - \frac{q'}{q^2} = 0 \quad \Longrightarrow \quad p' = -\frac{p^2}{q^2}q'$$

    b) At a certain moment, $p(t_0) = \dfrac{4}{3}$ and $p'(t_0) = 2$. Find $q(t_0)$ and $q'(t_0)$.

$$\frac{1}{p(t_0)} + \frac{1}{q(t_0)} = 1 \quad \Longrightarrow \quad \frac{1}{\dfrac{4}{3}} + \frac{1}{q(t_0)} = 1 \quad \Longrightarrow \quad q(t_0) = 4$$

$$q' = -\frac{q^2}{p^2}p' = -\frac{4^2}{\left(\dfrac{4}{3}\right)^2} \cdot 2 = -18$$

11. The base radius and height of a cylinder are constantly changing but the volume of the cylinder is kept at a constant 600π in^3.

   a) At a time t_1 the base radius is $r(t_1) = 10$ in and its rate of change is $r'(t_1) = 0.2\dfrac{\text{in}}{\text{s}}$. Compute the rate of change of the height of the cylinder $h'(t)$ at time t_1.

   Solution: $V = \pi r^2 h \qquad h = \dfrac{V}{\pi r^2}$

   $$\begin{aligned} V &= \pi r^2 h \\ 0 &= \pi\left(2rr'h + r^2 h'\right) \\ 0 &= 2rr'h + r^2 h' \\ \frac{-2rr'h}{r^2} &= h' \end{aligned}$$

   $$h' = \frac{-2r'h}{r} = \frac{-2r'\left(\dfrac{V}{\pi r^2}\right)}{r} = \frac{-2r'V}{\pi r^3} = \frac{-2\left(0.2\dfrac{\text{in}}{\text{s}}\right)\left(600\pi\,\text{in}^3\right)}{\pi\left(10\,\text{in}\right)^3} = -0.24\frac{\text{in}}{\text{s}}$$

   b) At a time t_2 the height is $h(t_2) = 12$ in and its rate of change is $r'(t_2) = -0.5\dfrac{\text{in}}{\text{s}}$. Compute the rate of change of the radius of the cylinder $r(t)$ at time t_2.

   Solution: $V = \pi r^2 h \qquad r = \sqrt{\dfrac{V}{\pi h}}$

   $$\begin{aligned} V &= \pi r^2 h \\ 0 &= \pi\left(2rr'h + r^2 h'\right) \\ 0 &= 2rr'h + r^2 h' \end{aligned}$$

   $$r' = \frac{-r^2 h'}{2rh} = \frac{-rh'}{2h} = \frac{-\left(\sqrt{\dfrac{V}{\pi h}}\right)h'}{2h} = \frac{-\left(\sqrt{\dfrac{600\pi\,\text{in}^3}{\pi\,(12\,\text{in})}}\right)\left(-0.5\dfrac{\text{in}}{\text{s}}\right)}{2\,(12\,\text{in})} = \frac{-\left(\sqrt{50\,\text{in}^2}\right)\left(-0.5\dfrac{\text{in}}{\text{s}}\right)}{24\,\text{in}}$$

   $$= \frac{\left(5\sqrt{2}\,\text{in}\right)\left(0.5\dfrac{\text{in}}{\text{s}}\right)}{24\,\text{in}} = \frac{5}{48}\sqrt{2}\frac{\text{in}}{\text{s}} \approx 0.147\,314\frac{\text{in}}{\text{s}}$$

12. An object, dropped from a height of h has a location of $y(t) = -16t^2 + h$ feet after t seconds. We dropped a small object from a height of 60 feet.
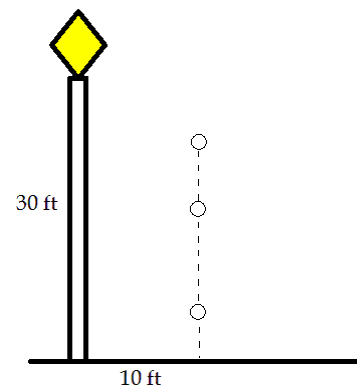
   a) Where is the object and what is its velocity after 1.5 seconds?

   Solution: we substitute $t = 1.5$ into the formula $y(t) = -16t^2 + 60$ and get $y(1.5\,\text{s}) = 24\,\text{ft}$

   For the velocity, we differentiate $y(t)$ with resepct to t and evaluate the derivative at $t = 1.5\,\text{s}$.

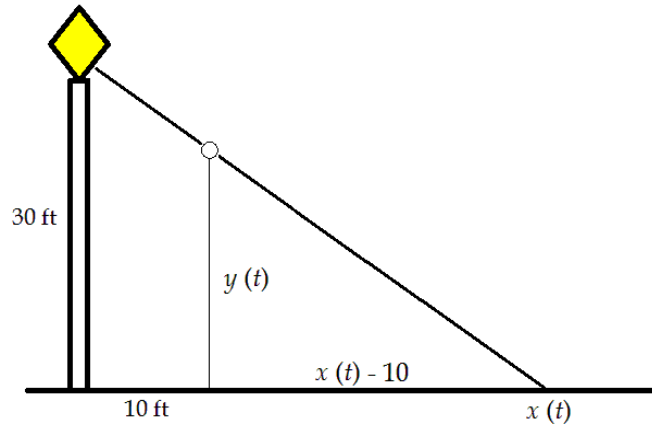 $$y'(t) = -32t \qquad y'(1.5) = -32\,(1.5) = -48$$

   So $y'(1.5\,\text{s}) = -48\dfrac{\text{ft}}{\text{s}}$

b) Suppose there is a 30 feet tall street light 10 feet away from the point where the object will land. How far is the shadow of the object from the base of the street light at $t = 1.5$?

Solution: Let us denote by $y(t)$ the vertical position of the object and by $x(t)$ the horizontal position of its shadow.

By similar triangles, we have that

$$\frac{y(t)}{x(t) - 10} = \frac{30}{x(t)} \quad \text{clear denominators}$$
$$x(t)\, y(t) = 30\,(x(t) - 10)$$

We use this equation first to solve for $x(t)$

$$x(t)\, y(t) = 30x(t) - 300$$
$$300 = 30x(t) - x(t)\, y(t)$$
$$300 = x(t)\,(30 - y(t))$$
$$\frac{300}{30 - y(t)} = x(t)$$

In particular, $x(1.5) = \dfrac{300}{30 - y(1.5)} = \dfrac{300}{30 - 24} = 50$.

c) How fast is the obejct's shadow moving at $t = 1.5$?

Solution: To compute $x'(1.5)$, we differentiate $x(t)\, y(t) = 30\,(x(t) - 10)$

$$y(t)\, x(t) = 30\,(x(t) - 10)$$
$$x'(t)\, y(t) + x(t)\, y'(t) = 30x'(t) \qquad \text{Solve for } x'(t)$$
$$x(t)\, y'(t) = 30x'(t) - x'(t)\, y(t)$$
$$x(t)\, y'(t) = x'(t)\,(30 - y(t))$$
$$\frac{x(t)\, y'(t)}{30 - y(t)} = x'(t)$$

So $x'(1.5)$ is

$$x'(1.5) = \frac{(50\,\text{ft})\left(-48\dfrac{\text{ft}}{\text{s}}\right)}{30\,\text{ft} - 24\,\text{ft}} = -400\frac{\text{ft}}{\text{s}}$$

The negative sign indicates that the shadow is traveling toward the left.

13. Two boats leave a port at the same time, one traveling west at 20 mi/hr and the other traveling southwest at 15 mi/hr. At what rate is the distance between them changing 30 min after they leave the port?

Solution: Let (x_1, y_1) denote the location of the car moving to the west. Let (x_2, y_2) denote the location of the car moving southwest.
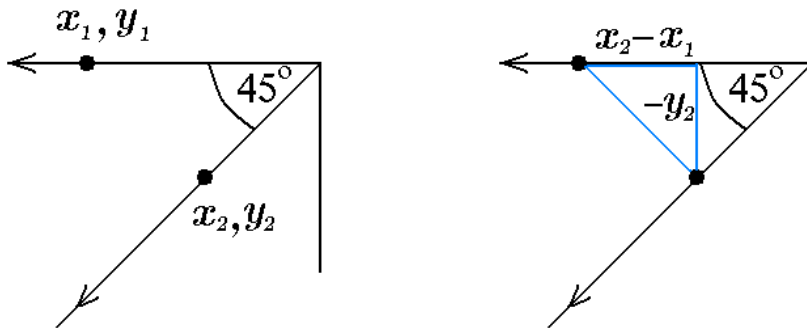
Then $x_1(t) = -20t$, $y_1(t) = 0$ and $x_2(t) = -\dfrac{15}{\sqrt{2}}t$ and $y_2(t) = -\dfrac{15}{\sqrt{2}}t$. And then $\dfrac{dx_1}{dt} = -20$, $\dfrac{dy_1}{dt} = 0$,

$\dfrac{dx_2}{dt} = -\dfrac{15}{\sqrt{2}} = -\dfrac{15\sqrt{2}}{2}$ and $\dfrac{dy_2}{dt} = -\dfrac{15}{\sqrt{2}} = -\dfrac{15\sqrt{2}}{2}$.

First we will express the distance between the cars. For that, we will use the Pythagorean theorem.

$$s^2 = (x_2 - x_1)^2 + (-y_2)^2$$

We differentiate both sides with respect to t and solve for $\dfrac{ds}{dt}$

$$2s\frac{ds}{dt} = 2(x_2 - x_1)\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + 2(-y_2)\left(-\frac{dy_2}{dt}\right)$$

$$\frac{ds}{dt} = \frac{2(x_2 - x_1)\left(\dfrac{dx_2}{dt} - \dfrac{dx_1}{dt}\right) + 2y_2\left(\dfrac{dy_2}{dt}\right)}{2s}$$

$$\frac{ds}{dt} = \frac{(x_2 - x_1)\left(\dfrac{dx_2}{dt} - \dfrac{dx_1}{dt}\right) + y_2\left(\dfrac{dy_2}{dt}\right)}{\sqrt{(x_2 - x_1)^2 + (-y_2)^2}}$$

After 30 minutes, $x_1 = -10$ $y_1 = 0$ and $x_2 = -\dfrac{15}{\sqrt{2}}\left(\dfrac{1}{2}\right) = -\dfrac{15}{4}\sqrt{2}$ and $y_2(t) = -\dfrac{15}{\sqrt{2}}\left(\dfrac{1}{2}\right) = -\dfrac{15\sqrt{2}}{4}$. At that time,

$$\frac{ds}{dt} = \frac{(x_2 - x_1)\left(\dfrac{dx_2}{dt} - \dfrac{dx_1}{dt}\right) + y_2\left(\dfrac{dy_2}{dt}\right)}{\sqrt{(x_2 - x_1)^2 + (-y_2)^2}}$$

$$\frac{ds}{dt} = \frac{(x_2 - x_1)\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + y_2\left(\frac{dy_2}{dt}\right)}{\sqrt{(x_2 - x_1)^2 + (-y_2)^2}}$$

$$\frac{ds}{dt} = \frac{\left(-\frac{15}{4}\sqrt{2} - (-10)\right)\left(-\frac{15\sqrt{2}}{2} - (-20)\right) + \left(-\frac{15\sqrt{2}}{4}\right)\left(-\frac{15}{\sqrt{2}}\right)}{\sqrt{\left(-\frac{15}{4}\sqrt{2} - (-10)\right)^2 + \left(\frac{15\sqrt{2}}{4}\right)^2}}$$

$$= \frac{\left(-3.75\sqrt{2} + 10\right)\left(-7.5\sqrt{2} + 20\right) + \frac{225}{4}}{\sqrt{\left(-3.75\sqrt{2} + 10\right)^2 + 2\left(\frac{15}{4}\right)^2}} \approx 14.168\,131$$

Or:

Let a be the disance from the origin of one car and b the distance of the origin from the other car. Then

$$s^2 = a^2 + b^2 - 2ab\cos 45°$$

we differentiate both sides:

$$2ss' = 2aa' + 2bb' - 2\left(\frac{\sqrt{2}}{2}\right)(a'b + ab')$$

$$s' = \frac{aa' + bb' - \left(\frac{\sqrt{2}}{2}\right)(a'b + ab')}{s} =$$

$$= \frac{10\,(20) + 7.5\,(15) - \left(\frac{\sqrt{2}}{2}\right)(20\,(7.5) + 10\,(15))}{\sqrt{10^2 + 7.5^2 - 2\,(10)\,(7.5)\cos 45°}} \approx 14.168\,131$$

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