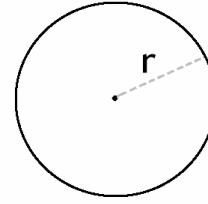


Definition: A circle is the set of all points in a plane that are equidistant to a fixed point. That equal distance is called the radius of the circle, that fixed point is called the center of the circle.

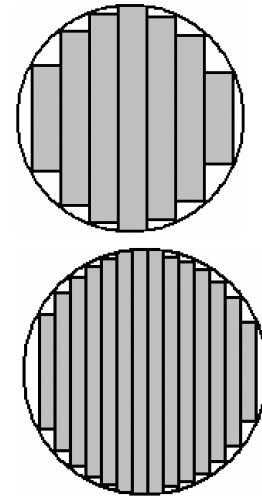


Theorem: The circumference of a circle with radius r is $C = 2\pi r$.
The area of a circle with radius r is $A = \pi r^2$.

The letter π (read as 'pie') is the lower case letter 'p' in the Greek alphabet. It represents a number that is associated with the area and circumference of circles. It is a tricky number with strange and interesting properties, and an unpleasant decimal form. $\pi \approx 3.14159365\dots$. As it turns out, these decimals go on forever. Worse yet, the decimals do not repeat like they do in the decimal presentation of $\frac{1}{3}$. So, if we want to be precise without writing tediously long decimals, it is practical to just use the symbol π . We will refer to this symbolic presentation as the exact value, as opposed to approximate values. Writing down the character π doesn't take up a lot of space or time, and the reader can use their calculator to find the answer in a decimal form, up to the precision they need.

Most calculators have a π button. Whenever it is possible, it should be used instead of entering 3.14. The button carries a much more precise approximation.

At this level of mathematics, we do not have the tools necessary to prove these formulas. The proof actually uses (again) the formula for the area of a rectangle. The basic idea is to first find an estimation for the area using rectangles.



We can compute the grey area since it is composed of rectangles. But of course this is a crude underestimation of the actual area. The trick is that the more rectangles we use, the more accurate the approximation becomes. Using more and more rectangles to estimate the area of the semi-circle, the approximations become better and better.

Calculus offers tools to find a unique number these approximations approach. This number is defined to be the area of the circle.

Example 1. Find the circumference and area of a circle of radius 7 ft.

Solution: We apply the formulas: $C = 2\pi r$ and $A = \pi r^2$. We get better precision if we use the π button on the calculator instead of entering 3.14.

$$C = 2\pi r^2 = 2\pi (7 \text{ ft}) = 14\pi \text{ ft} \approx 43.9823$$

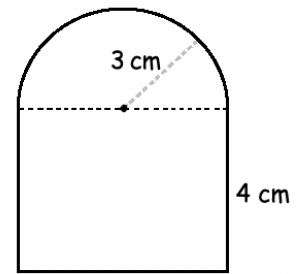
and

$$A = \pi r^2 = \pi (7\text{ft})^2 = 49\pi \text{ ft}^2 \approx 153.93804 \text{ ft}^2$$

The exact values are $14\pi \text{ ft}$ and $49\pi \text{ ft}^2$. It is nice to present these to the reader in case they will need a better precision than that in our approximate values, 43.9823 ft and 153.93804 ft².

Example 2. Find the area of the figure shown on the picture.

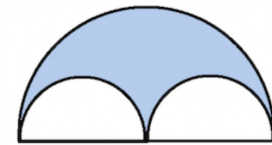
Solution: The area is the sum of the areas of a rectangle and a semi-circle. What is interesting in this problem is that we need our geometric intuition to find the horizontal side of the rectangle. Because they align, we can conclude that the horizontal side is the same as the diameter of the circle. (The diameter d is the longest line segment that can be drawn inside, in case of a circle it is simply twice the radius).



We can now apply the appropriate formulas and then add the results. Because the top is a semicircle, we need to take half of both the area and circumference of the circle. The horizontal side of the rectangle is 6 cm since we can fit exactly two radii on it.

$$\begin{aligned} A_{\text{rectangle}} &= ab = 4 \text{ cm} (6 \text{ cm}) = 24 \text{ cm}^2 \\ A_{\text{semicircle}} &= \frac{\pi r^2}{2} = \frac{\pi (3 \text{ cm})^2}{2} \approx 14.137 \text{ cm}^2 \\ A &= A_{\text{rectangle}} + A_{\text{semicircle}} \approx 24 \text{ cm}^2 + 14.137 \text{ cm}^2 = 38.137 \text{ cm}^2 \end{aligned}$$

Example 3. The picture shows three semi-circles. The smaller semi-circles both have a radius of 6 cm. Find the area of the shaded region.



Solution: The shaded area can be found by computing the area of the big semi-circle and subtracting the areas of the smaller semi-circles.

The tricky part is that the radius of the larger semi-circle is not given. We have to figure that one out from the geometric relationships shown on the picture.

The diameter of the larger semi-circle is as long as the two diameters of the smaller semi-circles. So the large semi-circle's diameter is 24 cm and thus its radius is 12 cm. We are now ready to compute the area.

The area of the big semi-circle is:

$$A_{\text{big}} = \frac{\pi (12 \text{ cm})^2}{2} = \frac{144\pi \text{ cm}^2}{2} = 72\pi \text{ cm}^2 \approx 226.1947 \text{ cm}^2$$

The area of each of the smaller semi-circle is:

$$A_{\text{small}} = \frac{\pi (6 \text{ cm})^2}{2} = \frac{36\pi \text{ cm}^2}{2} = 18\pi \text{ cm}^2 \approx 56.5487 \text{ cm}^2$$

The area of each of the two smaller semi-circles are:

$$A_{\text{two semi-circles}} = 2 \cdot 18\pi \text{ cm}^2 \approx 113.0973 \text{ cm}^2$$

So the shaded area is

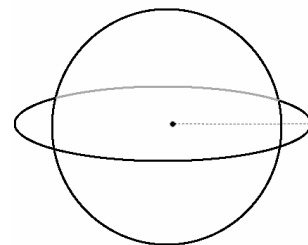
$$A \approx 226.1947 \text{ cm}^2 - 113.0973 \text{ cm}^2 = 113.0973 \text{ cm}^2$$

Example 4. A satellite is always above the Earth's equator. It revolves around Earth at a height of 600 miles. The radius of the Earth is 3960 miles long. How many miles long is the satellite's circular orbit? Round your answer to the nearest mile.

Solution: The orbit of the satellite is a circle with a radius longer than that of the Earth. Its radius is in fact the sum of the radius of Earth and the height. So
 $r = 3960 \text{ mi} + 600 \text{ mi} = 4560 \text{ mi}$. The rest of the problem is easy, we just need to find the circumference of the circle with this radius.

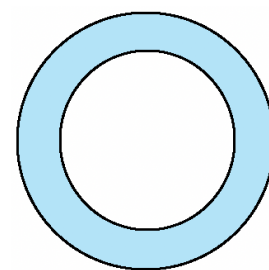
$$C = 2\pi r = 2\pi (4560 \text{ mi}) = 9120\pi \text{ mi} \approx 28\,561.325 \text{ mi}$$

When rounding to the nearest mile, we get 28 651 miles.



Example 5. Find the area of the shaded region shown on the picture. The radii of the two circles are 5 ft and 4 ft.

Solution: To find the area of the ring, we will compute the area of the bigger circle, the area of the smaller circle, and then we subtract.

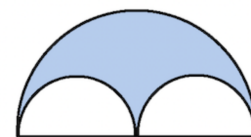
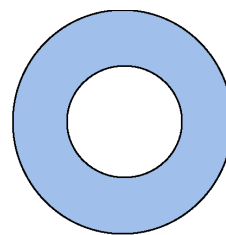


$$\begin{aligned} A_{\text{big}} &= \pi r_{\text{big}}^2 = \pi (5 \text{ ft})^2 = 25\pi \text{ ft}^2 \\ A_{\text{small}} &= \pi r_{\text{small}}^2 = \pi (4 \text{ ft})^2 = 16\pi \text{ ft}^2 \\ A &= A_{\text{big}} - A_{\text{small}} = 25\pi \text{ ft}^2 - 16\pi \text{ ft}^2 = 9\pi \text{ ft}^2 \approx 28.274334 \text{ ft}^2 \end{aligned}$$

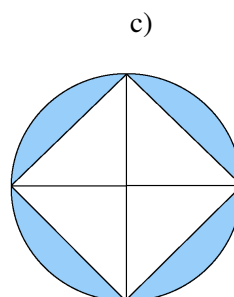
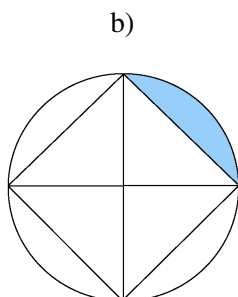
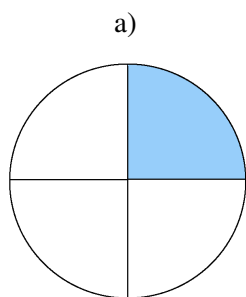


Practice Problems

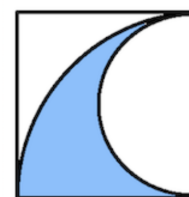
1. Find the area of the ring shown on the picture. The circles have the same center and the radii are 3 m and 4 m.
2. Suppose that every pizza we make today will have the same thickness. What contains then more calories, a 12-inch-diameter pizza or two 6-inch-diameter pizzas?
3. The picture shows three semicircles. The radius of both of the smaller semicircles is 2 m. Find the area of the shaded region.



4. All three circles shown below have a radius of 10 units. In each case, find the approximate value of the area of the shaded region.



5. A square, a quarter circle, and a semi-circle. Find the area of the shaded region if the square has sides 20 cm long.





Answers

1. 21.991149 m^2

2. (pizzas) Comparing the areas, two 3-inch radius pizzas will have a combined area of

$$A = 2\pi r^2 = 2\pi (3 \text{ in})^2 = 2 \cdot 9\pi \text{ in}^2 = 18\pi \text{ in}^2 \approx 56.54867 \text{ in}^2$$

and one 6-inch radius pizza will have an area of

$$A = \pi R^2 = \pi (6 \text{ in})^2 = 36\pi \text{ in}^2 \approx 113.097 \text{ in}^2$$

and so the 12-diameter pizza will contain more calories.

3. $4\pi \text{ m}^2 \approx 12.5664 \text{ m}^2$

4. a) 78.5398 unit^2 b) 28.5398 unit^2 c) 114.159 unit^2

5. 157.0796 cm^2