Equations are a fundamental concept and tool in mathematics.

Definition: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign.

For example, $3x^2 - x = 4x + 28$ is an equation. So is $x^2 + 5y = -y^2 + x + 2$.

Definition: A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality true.

Example 1: a) Verify that -2 is not a solution of the equation $3x^2 - x = 4x + 28$.

b) Verify that 4 is a solution of the equation $3x^2 - x = 4x + 28$.

c) Verify that the pair of numbers $x = 3$ and $y = -4$ is a solution of the equation $x^2 + 5y = -y^2 + x + 2$.

d) Verify that the pair of numbers $x = -4$ and $y = 3$ is not a solution of the equation $x^2 + 5y = -y^2 + x + 2$.

Solution: a) Consider the equation $3x^2 - x = 4x + 28$ with $x = -2$.
If $x = -2$, the left-hand side of the equation is $\text{LHS} = 3x^2 - x = 3(-2)^2 - (-2) = 3 \cdot 4 + 2 = 12 + 2 = 14$ and the right-hand side is $\text{RHS} = 4x + 28 = 4(-2) + 28 = -8 + 28 = 20$
Since the two sides are not equal, $14 \neq 20$, the number -2 is not a solution of this equation.

b) Consider the equation $3x^2 - x = 4x + 28$ with $x = 4$.
If $x = 4$, the left-hand side of the equation is $\text{LHS} = 3x^2 - x = 3 \cdot 4^2 - 4 = 3 \cdot 16 - 4 = 48 - 4 = 44$ and the right-hand side is $\text{RHS} = 4x + 28 = 4 \cdot 4 + 28 = 16 + 28 = 44$
Since the two sides are equal, $x = 4$ is a solution of this equation.

c) Consider the equation $x^2 + 5y = -y^2 + x + 2$ with $x = 3$ and $y = -4$.
If $x = 3$ and $y = -4$, the left-hand side of the equation is $\text{LHS} = x^2 + 5y = 3^2 + 5(-4) = 9 - 20 = -11$ and the right-hand side is $\text{RHS} = -y^2 + x + 2 = -(-4)^2 + 3 + 2 = -16 + 3 + 2 = -11$
Since the two sides are equal, $x = 3$ and $y = -4$ is a solution of this equation.

d) Consider the equation $x^2 + 5y = -y^2 + x + 2$ with $x = -4$ and $y = 3$.
If $x = -4$ and $y = 3$, the left-hand side of the equation is $\text{LHS} = x^2 + 5y = (-4)^2 + 5 \cdot 3 = 16 + 15 = 41$ and the right-hand side is $\text{RHS} = -y^2 + x + 2 = -3^2 + (-4) + 2 = -9 - 4 + 2 = -11$
Since $41 \neq -11$, the two sides are not equal, and so $x = -4$ and $y = 3$ is not a solution of this equation.

If the equation is in more than one variable, like in parts c) and d) before, it is important to identify which number is to be substituted into which variable. After all, $x = 3$ and $y = -4$ (or, as an *ordered pair*, $(3, -4)$) is a solution of the equation $x^2 + 5y = -y^2 + x + 2$, but $x = -4$ and $y = 3$, (or, as an *ordered pair*, $(-4, 3)$) is not.

Definition: To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Caution! Finding one solution for an equation is not the same as solving it. For example, we found that 4 is a solution of $3x^2 - x = 4x + 28$. As it turns out, 4 is not the only solution. We leave to the reader to verify that $-\dfrac{7}{3}$ is also a solution of the equation. We will have to deploy systematic methods to find all solutions. The methods we will use usually depends on the type of equation. We will start with the simplest equations, linear equations.

Linear equations are a fundamental concept and tool in mathematics. To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides.

Example 2: Solve each of the given equations. Make sure to check your solutions.

a) $x - 8 = 10$ b) $3y = -12$ c) $x - \frac{1}{6} = \frac{2}{3}$ d) $m + 10 = -5$

Equations like these are called **one-step equations** because they can be solved in only one step. We need to isolate the unknown on one side. In order to do that, we perform the inverse operation. (The inverse operation of additon is subtraction and vica versa. The inverse operation of multiplication is division and vica versa.)

Solution: a) In order to isolate the unknown, we add 8 to both sides.

$$\begin{aligned} x - 8 &= 10 \qquad \text{add } 8 \\ x &= 18 \end{aligned}$$

So the only solution of this equation is 18. We can also say that the solution set is $\{18\}$. We should check; if $x = 18$, the left-hand side is

$$\text{LHS} = x - 8 = 18 - 8 = 10 = \text{RHS}$$

So our solution, $\boxed{x = 18}$ is correct.

b) In order to isolate the unknown, we divide both sides by 3.

$$\begin{aligned} 3y &= -12 \qquad \text{divide by } 3 \\ y &= -4 \end{aligned}$$

So the only solution of this equation is -4. We check; if $y = -4$, then

$$\text{LHS} = 3y = 3(-4) = -12 = \text{RHS}$$

So our solution, $\boxed{y = -4}$ is correct.

c) In order to isolate the unknown, we add $\frac{1}{6}$ to both sides.

$$\begin{aligned} x - \frac{1}{6} &= \frac{2}{3} \qquad \text{add } \frac{1}{6} \qquad\qquad \text{margin work: } \frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6} \\ x &= \frac{5}{6} \end{aligned}$$

So the only solution of this equation is $\frac{5}{6}$. We check; if $x = \frac{5}{6}$, then

$$\text{LHS} = x - \frac{1}{6} = \frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3} = \text{RHS}$$

So our solution, $\boxed{x = \frac{5}{6}}$ is correct.

d) In order to isolate the unknown, we we subtract 10 from both sides.

$$\begin{aligned} m + 10 &= -5 \qquad \text{subtract } 10 \\ m &= -15 \end{aligned}$$

So the only solution of this equation is -15. We check; if $m = -15$, then

$$\text{LHS} = m + 10 = -15 + 10 = -5 = \text{RHS}$$

So our solution, $\boxed{m = -15}$ is correct.

Note: If the reader is interested in applications of one-step equations, basic percent problems and basic motion problems can be easily handled by setting up and solving one-step equations.

Discussion: Solve each of the following equations. How are these unusual?

a) $5x = 5$ b) $5x = 0$ c) $x - 4 = -4$ d) $\frac{x}{3} = 0$ e) $\frac{1}{3}x = 0$

Example 3: Solve each of the given equations. Make sure to check your solutions.

a) $10 = 3x - 11$ b) $\frac{1}{2}x + 1 = -7$ c) $\frac{t-7}{2} = -8$ d) $\frac{x}{-3} + 4 = 15$

Equations like these are called **two-step equations**. We need to isolate the unknown on one side. In order to do that, we perform the inverse operations, in the reverse order it was done to the unknown.

Solution: a) This equation looks unusual in the sense that two-step equations often contain the unknown on the left-hand side. We are always allowed to swap two sides of an equation. If $A = B$, then clearly, also $B = A$. We will do this first. Notice that this is an optional step.

$$\begin{aligned} 10 &= 3x - 11 && \text{swap the two sides} \\ 3x - 11 &= 10 \end{aligned}$$

We now look at the side that contains x and ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and subtraction of* 11. We need to apply the inverse operations, in a reverse order. In this case, this means that we will add 11 to both sides and then divide both sides by 3.

$$\begin{aligned} 3x - 11 &= 10 && \text{add } 11 \\ 3x &= 21 && \text{divide by } 3 \\ x &= 7 \end{aligned}$$

So the only solution of this equation is 7. We check: if $x = 7$, then

$$\text{LHS} = 3x - 11 = 3 \cdot 7 - 11 = 21 - 11 = 10 = \text{RHS}$$

So our solution, $\boxed{x = 7}$ is correct.

b) As we look at the left-hand side and ask: *What happened to the unknown*? The answer is: *Multiplication by* $\frac{1}{2}$ *and addition of* 1. We need to apply the inverse operations, in a reverse order. In this case, this means that we will subtract 1 from both sides and then divide both sides by $\frac{1}{2}$.

$$\begin{aligned} \frac{1}{2}x + 1 &= -7 && \text{subtract } 1 \\ \frac{1}{2}x &= -8 && \text{divide by } \frac{1}{2} \\ x &= -16 \end{aligned}$$

margin work: to divide is to multiply by the reciprocal: $\frac{-8}{\frac{1}{2}} = -8 \cdot 2 = -16$

So the only solution of this equation is -16. We check: if $x = -16$, then

$$\text{LHS} = \frac{1}{2}x + 1 = \frac{1}{2}(-16) + 1 = -8 + 1 = -7 = \text{RHS}$$

So our solution, $\boxed{x = -16}$ is correct.

c) What happened to the unknown? On the left-hand side, there was a subtraction of 7 and then a division by 2. To reverse that, we will multiply both sides by 2 and then add 7 to both sides.

$$\begin{aligned} \frac{t-7}{2} &= -8 && \text{multiply by } 2 \\ t - 7 &= -16 && \text{add } 7 \\ t &= -9 \end{aligned}$$

So the only solution of this equation is -9. We check: if $t = -9$, then

$$\text{LHS} = \frac{t-7}{2} = \frac{-9-7}{2} = \frac{-16}{2} = -8 = \text{RHS}$$

So our solution, $\boxed{t = -9}$ is correct.

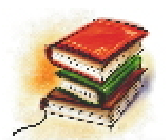
d) What happened to the unknown? On the left-hand side, there was a division by -3 and then an addition of 4. To reverse that, we will subtract 4 from both sides by and then multiply both sides by -3.

$$\frac{x}{-3} + 4 = 15 \qquad \text{subtract } 4$$
$$\frac{x}{-3} = 11 \qquad \text{multiply by } -3$$
$$x = -33$$

So the only solution of this equation is -33. We check: if $x = -33$, then

$$\text{LHS} = \frac{x}{-3} + 4 = \frac{-33}{-3} + 4 = 11 + 4 = 15 = \text{RHS}$$

So our solution, $\boxed{x = -33}$ is correct.



Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x - 5 = 17$
2. $\dfrac{a - 10}{5} = -3$
3. $\dfrac{t}{4} - 10 = -4$
4. $\dfrac{t - 5}{12} = 4$
5. $2x - 7 = -3$
6. $\dfrac{x + 8}{3} = -2$
7. $\dfrac{x}{3} + 8 = -2$
8. $-2x + 3 = 3$
9. $3(x + 7) = 36$
10. $\dfrac{1}{5}x - \dfrac{2}{3} = \dfrac{26}{15}$
11. $\dfrac{3}{8}x + \left(1\dfrac{4}{5}\right) = \dfrac{3}{10}$

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x - 3 = -11$
2. $-2x - 3 = 7$
3. $5x - 3 = 17$
4. $\dfrac{x - 3}{7} = -2$
5. $\dfrac{x}{7} - 3 = -1$
6. $-4x - 3 = 13$
7. $\dfrac{a + 1}{4} = -9$
8. $5x - 6 = -6$
9. $\dfrac{x}{7} - 1 = -3$
10. $-x + 5 = -7$
11. $\dfrac{2x - 1}{7} = -3$
12. $5(x - 2) = -20$
13. $\dfrac{x + \frac{3}{8}}{2\frac{4}{5}} = \dfrac{5}{16}$
14. $\dfrac{2}{3}b + \dfrac{3}{5} = -\dfrac{1}{15}$
15. $\dfrac{1}{3}x + \dfrac{2}{5} = -\dfrac{34}{15}$

Answers

Discussion

a) 1 b) 0 c) 0 d) 0 e) 0

One thing that is unusual in this problem is the idea of cancellation. Cancellation results in 0 or 1, depending on the operation.

Sample Problems

1.) 11 2.) -5 3.) 24 4.) 53 5.) 2 6.) -14 7.) -30 8.) 0 9.) 5 10.) 12 11.) -4

Practice Problems

1.) -4 2.) -5 3.) 4 4.) -11 5.) 14 6.) -4 7.) -37 8.) 0 9.) -14 10.) 12

11.) -10 12.) -2 13.) $\frac{1}{2}$ 14.) -1 15.) -8

Sample Problems - Solutions

Solve each of the following equations. Make sure to check your solutions.

1. $2x - 5 = 17$

   Solution:

$$\begin{aligned} 2x - 5 &= 17 && \text{add 5 to both sides} \\ 2x &= 22 && \text{divide by 2} \\ x &= 11 \end{aligned}$$

   We check: if $x = 11$, then

$$\text{RHS} = 2(11) - 5 = 22 - 5 = 17 = \text{LHS}$$

   Thus our solution, $\boxed{x = 11}$ is correct.

2. $\dfrac{a - 10}{5} = -3$

   Solution:

$$\begin{aligned} \frac{a - 10}{5} &= -3 && \text{multiply both sides by 5} \\ a - 10 &= -15 && \text{add 10 to both sides} \\ a &= -5 \end{aligned}$$

   We check: if $a = -5$, then

$$\text{LHS} = \frac{-5 - 10}{5} = \frac{-15}{5} = -3 = \text{RHS}$$

   Thus our solution, $\boxed{a = -5}$ is correct.

3. $\dfrac{t}{4} - 10 = -4$

   Solution:

$$\begin{aligned} \frac{t}{4} - 10 &= -4 && \text{add } 10 \text{ to both sides} \\ \frac{t}{4} &= 6 && \text{multiply both sides by } 4 \\ t &= 24 \end{aligned}$$

   We check: if $t = 24$, then

$$\text{RHS} = \frac{t}{4} - 10 = \frac{24}{4} - 10 = 6 - 10 = -4 = \text{LHS}$$

   Thus our solution, $\boxed{t = 24}$ is correct.

4. $\dfrac{t-5}{12} = 4$

   Solution:

$$\begin{aligned} \frac{t-5}{12} &= 4 && \text{multiply both sides by } 12 \\ t - 5 &= 48 && \text{add } 5 \text{ to both sides} \\ t &= 53 \end{aligned}$$

   We check: if $t = 53$, then

$$\text{RHS} = \frac{53-5}{12} = \frac{48}{12} = 4 = \text{LHS}$$

   Thus our solution, $\boxed{t = 53}$ is correct.

5. $2x - 7 = -3$

   Solution: We apply all operations to both sides.

$$\begin{aligned} 2x - 7 &= -3 && \text{add } 7 \\ 2x &= 4 && \text{divide by } 2 \\ x &= 2 \end{aligned}$$

   We check: if $x = 2$, then

$$\text{LHS} = 2(2) - 7 = 4 - 7 = -3 = \text{RHS}$$

   Thus our solution, $\boxed{x = 2}$ is correct.

6. $\dfrac{x+8}{3} = -2$

   Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x+8}{3} &= -2 && \text{multiply by } 3 \\ x + 8 &= -6 && \text{subtract } 8 \\ x &= -14 \end{aligned}$$

   We check:

$$\text{LHS} = \frac{-14+8}{3} = \frac{-6}{3} = -2 = \text{RHS}$$

   Thus our solution, $\boxed{x = -14}$ is correct.

7. $\dfrac{x}{3}+8=-2$

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x}{3}+8 &= -2 && \text{subtract } 8 \\ \frac{x}{3} &= -10 && \text{multiply by } 3 \\ x &= -30 \end{aligned}$$

We check:

$$\text{LHS} = \frac{-30}{3}+8 = -10+8 = -2 = \text{RHS}$$

Thus our solution, $\boxed{x=-30}$ is correct.

8. $-2x+3=3$

Solution: We apply all operations to both sides.

$$\begin{aligned} -2x+3 &= 3 && \text{subtract } 3 \\ -2x &= 0 && \text{divide by } -2 \\ x &= 0 \end{aligned}$$

We check: if $x=0$, then

$$\text{LHS} = -2\cdot 0+3 = 0+3 = 3 = \text{RHS}$$

Thus our solution, $\boxed{x=0}$ is correct.

9. $3\,(x+7)=36$

Solution: We apply all operation to both sides,

$$\begin{aligned} 3\,(x+7) &= 36 && \text{divide by } 3 \\ x+7 &= 12 && \text{subtract } 7 \\ x &= 5 \end{aligned}$$

We check: if $x=5$, then

$$\text{LHS} = 3\,(5+7) = 3\cdot 12 = 36 = \text{RHS}$$

Thus our solution, $\boxed{x=5}$ is correct.

10. $\dfrac{1}{5}x-\dfrac{2}{3}=\dfrac{26}{15}$

Solution:

$$\begin{aligned} \frac{1}{5}x-\frac{2}{3} &= \frac{26}{15} && \text{add } \frac{2}{3} \text{ to both sides} && \frac{26}{15}+\frac{2}{3}=\frac{26}{15}+\frac{2\cdot 5}{3\cdot 5}= \\ \frac{1}{5}x &= \frac{12}{5} && \text{divide by } \frac{1}{5} && \frac{26}{15}+\frac{10}{15}=\frac{36}{15}=\frac{\cancel{3}\cdot 12}{\cancel{3}\cdot 5}=\frac{12}{5} \\ x &= 12 && && \frac{\frac{12}{5}}{\frac{1}{5}}=\frac{12}{5}\cdot\frac{5}{1}=\frac{12\cdot\cancel{5}}{1\cdot\cancel{5}}=\frac{12}{1}=12 \end{aligned}$$

We check: if $x=12$, then

$$\begin{aligned} \text{LHS} &= \frac{1}{5}\cdot 12-\frac{2}{3}=\frac{1}{5}\cdot\frac{12}{1}-\frac{2}{3}=\frac{12}{5}-\frac{2}{3}=\frac{12\cdot 3}{5\cdot 3}-\frac{2\cdot 5}{3\cdot 5}=\frac{36}{15}-\frac{10}{15}=\frac{26}{15} \\ \text{RHS} &= \frac{26}{15} \end{aligned}$$

Thus our solution, $\boxed{x=12}$ is correct.

11. $\dfrac{3}{8}x + \left(1\dfrac{4}{5}\right) = \dfrac{3}{10}$

Solution: this is a very simple equation, much like $2x + 1 = 7$, only the numbers in it are fractions. However, the principles and operations regarding equations are the same.

$$\begin{aligned}
\frac{3}{8}x + \left(1\frac{4}{5}\right) &= \frac{3}{10} && \text{convert mixed number to improper fraction}\\
\frac{3}{8}x + \frac{9}{5} &= \frac{3}{10} && \text{subtract } \frac{9}{5} \text{ from both sides; } \frac{3}{10} - \frac{9}{5} = \frac{3}{10} - \frac{18}{10} = \frac{3-18}{10} = \frac{-15}{10} = \frac{-3}{2}\\
\frac{3}{8}x &= \frac{-3}{2} && \text{divide both sides by } \frac{3}{8}\\
x &= -4 && \left(\frac{-3}{2}\right) \div \left(\frac{3}{8}\right) = \frac{-3}{2}\cdot\frac{8}{3} = \frac{-24}{6} = -4
\end{aligned}$$

We check:

$$\text{RHS} = \frac{3}{8}(-4) + \left(1\frac{4}{5}\right) = \frac{3}{8}\cdot\frac{-4}{1} + \frac{9}{5} = \frac{-12}{8} + \frac{9}{5} = \frac{-3}{2} + \frac{9}{5} = \frac{-15}{10} + \frac{18}{10} = \frac{3}{10} = \text{LHS}$$

Thus our solution, $\boxed{x = -4}$ is correct.

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