

Definition: A **rational expression** is a quotient of two polynomials.

Rational expressions are abstract fractions, such as $\frac{1}{x}$, $\frac{x}{x+1}$, or $\frac{2x-5}{x^2-4x+3}$. Because they are fractions, we can do the same thing to rational expressions as we did with fractions formed from numbers. To present fractions as final results, we always use the reduced form in which numerator and denominator do not share any common factors.

Recall the fundamental property of fractions: the only thing we can do to a fraction without changing its value is multiplying or dividing both numerator and denominator by the same number. In order to bring rational expressions to lowest terms, we factor both numerator and denominator and cancel out the common factors. It is important to remember that we can only cancel out common factors; we can not cancel out anything unless both numerator and denominator are factored.

Example 1. Simplify each of the given rational expressions.

$$\text{a) } \frac{18ab^2cx}{12a^3b^2c^3x} \quad \text{b) } \frac{x^2-25}{x^2-6x+5} \quad \text{c) } \frac{2x^4+4x^3-6x^2}{4x^5-4x^4} \quad \text{d) } \frac{a-3}{3-a}$$

Solution: a) Between 18 and 12, the common factor is 6. Between a and a^3 , the common factor is a . Between c and c^3 , the common factor is c . The factors b^2 and x can be completely cancelled out.

$$\frac{18ab^2cx}{12a^3b^2c^3x} = \frac{6ab^2cx (3)}{6ab^2cx (2a^2c^2)} = \frac{\cancel{6ab^2cx} (3)}{\cancel{6ab^2cx} (2a^2c^2)} = \boxed{\frac{3}{2a^2c^2}}$$

b) We first factor both numerator and denominator. $x^2 - 25 = (x+5)(x-5)$ by the difference of squares theorem. We can factor the denominator by trial and error and get that $x^2 - 6x + 5 = (x-5)(x-1)$. Then we cancel out the common factors.

$$\frac{x^2-25}{x^2-6x+5} = \frac{(x+5)(x-5)}{(x-1)(x-5)} = \frac{(x+5)\cancel{(x-5)}}{(x-1)\cancel{(x-5)}} = \boxed{\frac{x+5}{x-1}}$$

c) We first factor both numerator and denominator and then cancel out factors common to both numerator and denominator. In both cases, we start with the greatest common factor or GCF.

$$\begin{aligned} 2x^4 + 4x^3 - 6x^2 &= 2x^2(x^2 + 2x - 3) = 2x^2(x+3)(x-1) \\ 4x^5 - 4x^4 &= 4x^4(x-1) \end{aligned}$$

The factors common to numerator and denominator are $2x^2$ and $x-1$. We can cancel those out.

$$\frac{2x^4+4x^3-6x^2}{4x^5-4x^4} = \frac{2x^2(x+3)(x-1)}{4x^4(x-1)} = \frac{2x^2(x+3)\cancel{(x-1)}}{4x^4\cancel{(x-1)}} = \boxed{\frac{x+3}{2x^2}}$$

d) This problem often presents a troubling situation because neither numerator nor denominator can be factored. But if we substitute a few values into a , we find that in each case, the result is -1 . The expressions $a-3$ and $3-a$ are opposites. We can factor out from one of the two and then we can cancel out the common linear factors.

$$\frac{a-3}{3-a} = \frac{a-3}{-1(-3+a)} = \frac{a-3}{-1(a-3)} = \frac{\cancel{(a-3)}}{-1\cancel{(a-3)}} = \frac{1}{-1} = \boxed{-1}$$

We can also perform operations on rational expressions. We can add and subtract them, just like we do with fractions but that will be a topic later. For now, we will perform much easier operations: multiplications and divisions. When we multiply two fractions, terms common to the numerator of one and the denominator of the other can also be cancelled out, just like in $\frac{5}{8} \cdot \frac{3}{10} = \frac{5}{8} \cdot \frac{3}{5 \cdot 2} = \frac{3}{8 \cdot 2} = \frac{3}{16}$.

Example 2. Perform the given operations and simplify.

$$\text{a) } \frac{2a^2 - 18}{3a^2 - 6a - 9} \cdot \frac{a^2 + 2a + 1}{6a^2 + 6a} \quad \text{b) } \frac{5y^2 + 5y}{8y - 16} \div \frac{10y^3 + 10y^2}{4y^2 - 16}$$

Solution: a) First we factor numerators and denominators in both rational expressions.

$$2a^2 - 18 = 2(a^2 - 9) = 2(a + 3)(a - 3)$$

$$3a^2 - 6a - 9 = 3(a^2 - 2a - 3) = 3(a - 3)(a + 1)$$

$$a^2 + 2a + 1 = (a + 1)(a + 1)$$

$$6a^2 + 6a = 6a(a + 1)$$

Now we are ready to perform the multiplication.

$$\begin{aligned} \frac{2a^2 - 18}{3a^2 - 6a - 9} \cdot \frac{a^2 + 2a + 1}{6a^2 + 6a} &= \frac{2(a + 3)(a - 3)}{3(a - 3)(a + 1)} \cdot \frac{(a + 1)(a + 1)}{6a(a + 1)} = \frac{2(a + 3)\cancel{(a - 3)}}{3\cancel{(a - 3)}(a + 1)} \cdot \frac{(a + 1)\cancel{(a + 1)}}{6a\cancel{(a + 1)}} \\ &= \frac{2(a + 3)(a + 1)}{3(6a)(a + 1)} = \frac{2(a + 3)\cancel{(a + 1)}}{3(6a)\cancel{(a + 1)}} = \frac{2(a + 3)}{3(6a)} \\ &= \frac{2(a + 3)}{3(2 \cdot 3a)} = \frac{\cancel{2}(a + 3)}{3(\cancel{2} \cdot 3a)} = \frac{a + 3}{3(3a)} = \boxed{\frac{a + 3}{9a}} \end{aligned}$$

f) This problem indicates a division. Recall that to divide is to multiply by the reciprocal. After we re-write the problem as multiplication of rational expressions, this problem becomes very similar to the previous one.

$$\frac{5y^2 + 5y}{8y - 16} \div \frac{10y^3 + 10y^2}{4y^2 - 16} = \frac{5y^2 + 5y}{8y - 16} \cdot \frac{4y^2 - 16}{10y^3 + 10y^2}$$

As before, we start with four factoring exercises.

$$5y^2 + 5y = 5y(y + 1) \quad \text{and} \quad 8y - 16 = 8(y - 2)$$

$$4y^2 - 16 = 4(y^2 - 4) = 4(y + 2)(y - 2) \quad \text{and} \quad 10y^3 + 10y^2 = 10y^2(y + 1)$$

$$\begin{aligned} \text{Therefore, } \frac{5y^2 + 5y}{8y - 16} \cdot \frac{4y^2 - 16}{10y^3 + 10y^2} &= \frac{5y(y + 1)}{8(y - 2)} \cdot \frac{4(y + 2)(y - 2)}{10y^2(y + 1)} = \frac{5y\cancel{(y + 1)}}{8(y - 2)} \cdot \frac{4(y + 2)\cancel{(y - 2)}}{10y^2\cancel{(y + 1)}} \\ &= \frac{5y}{8(y - 2)} \cdot \frac{4(y + 2)}{10y^2} = \frac{\cancel{5}y}{2 \cdot \cancel{4}} \cdot \frac{\cancel{4}(y + 2)}{\cancel{5} \cdot 2y \cdot y} = \boxed{\frac{y + 2}{4y}} \end{aligned}$$



Discussion: Consider the expression $\frac{a-3}{3-a}$ from example 1d). If we substitute a few values into a (say 2, -5 , 5, and 0), the value of the expression is -1 . This is not true for every number. Find a number that, when substituted into the expression, does not result in -1 .



Sample Problems

1. Simplify each of the following.

a) $\frac{2a-5}{5-2a}$ b) $\frac{x^3-x}{x+1}$ c) $\frac{2x+1}{4x^2-1}$ d) $\frac{x^2-4x+3}{x^2+2x-15}$ e) $\frac{(x+5)-2}{5(x+2)-(x-2)}$

2. Perform the indicated operations and simplify.

a) $\frac{c}{5a} \cdot \frac{15a^2b}{3b^2c}$ c) $\frac{x^2-3x}{x^2-8x+15} \cdot \frac{x^2-16x+15}{x^2-x}$ e) $\frac{x^2-10x+25}{x^2-10x+24} \left(\frac{x^2-2x-8}{x^2-6x+5} \div \frac{x-5}{x-1} \right)$
 b) $\frac{5x-30}{x^2-36} \cdot \frac{3x+18}{5}$ d) $\frac{x^2-9}{x^2-4x-21} \div \frac{4x-12}{3x-21}$ f) $\frac{6x^2y^2-36x^2y+48x^2}{py^2-2py-8p} \div \frac{-3y^2+24y-36}{p^2y+2p^2}$



Practice Problems

1. Simplify each of the following.

a) $\frac{2b-5}{10-4b}$ c) $\frac{4t^2-9}{4t-6}$ e) $\frac{3m+m^2-10}{11m+m^2+30}$ g) $\frac{-7x-11-3(x-2)}{5x-11+3(x+5)}$
 b) $\frac{x^2-1}{x+1}$ d) $\frac{p^2-p}{p^2-1}$ f) $\frac{x^2-x-2}{x^2-5x+6}$

2. Perform the indicated operations and simplify.

a) $\frac{x}{5yz} \cdot \frac{10x^2y^3z}{4xy^2}$ c) $\frac{x^2-5x}{x^2-2x-15} \cdot \frac{x^2-9}{x^2-3x}$ e) $\frac{5y-35}{y^2-2y-35} \cdot \frac{3y+15}{5y-5}$
 b) $\frac{a^2-8a+16}{a} \cdot \frac{a^3}{4-a}$ d) $\frac{x^2-4x-21}{x^2-49} \div \frac{8x+x^2+15}{2x+x^2-35}$ f) $\frac{2x^2-98}{x^2-6x-7} \div \frac{21+3x}{6x^2-6}$



Answers

Sample Problems

1. a) -1 b) $x^2 - x$ c) $\frac{1}{2x-1}$ d) $\frac{x-1}{x+5}$ e) $\frac{1}{4}$

2. a) $\frac{a}{b}$ b) 3 c) $\frac{x-15}{x-5}$ d) $\frac{3}{4}$ e) $\frac{x+2}{x-6}$ f) $\frac{-2px^2}{y-6}$

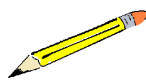
Practice Problems

1. a) $-\frac{1}{2}$ b) $x-1$ c) $\frac{2t+3}{2}$ d) $\frac{p}{p+1}$ e) $\frac{m-2}{m+6}$ f) $\frac{x+6}{x-8}$ g) $-\frac{5}{4}$

2. a) $\frac{x^2}{2}$ b) $-a^2(a-4)$ c) 1 d) $\frac{x-5}{x+5}$ e) $\frac{3}{y-1}$ f) $4x-4$

Discussion: If $a = 3$, the expression is undefined. For all other numbers, it is -1 .

Sample Problems



Solutions

1. Simplify each of the following.

a) $\frac{2a - 5}{5 - 2a}$

Solution: We need to notice that the numerator and denominator are opposites of each other. Indeed, the opposite of $2a - 5$ is $5 - 2a$ since

$$-1(2a - 5) = -2a + 5 = 5 - 2a$$

Thus

$$\frac{2a - 5}{5 - 2a} = \frac{2a - 5}{-1(2a - 5)} = \frac{\cancel{(2a - 5)}}{-1\cancel{(2a - 5)}} = \frac{1}{-1} = \boxed{-1}$$

b) $\frac{x^3 - x}{x + 1}$

Solution: In general, we factor both numerator and denominator and then simplify. In this case we only factor the numerator, since the denominator is too small to factor. After we factor out the greatest common factor (or GCF) which is x , the expression factors via the difference of squares theorem.

$$x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1)$$

Then we simplify the fraction by canceling out the same factor from numerator and denominator.

$$\frac{x^3 - x}{x + 1} = \frac{x(x + 1)(x - 1)}{x + 1} = \frac{\cancel{x}\cancel{(x + 1)}(x - 1)}{\cancel{x + 1}} = \boxed{x(x - 1) \text{ or } x^2 - x}$$

c) $\frac{2x + 1}{4x^2 - 1}$

Solution: We factor the denominator via the difference of squares theorem, and then cancel.

$$\frac{2x + 1}{4x^2 - 1} = \frac{2x + 1}{(2x + 1)(2x - 1)} = \frac{\cancel{2x + 1}}{\cancel{(2x + 1)}(2x - 1)} = \boxed{\frac{1}{2x - 1}}$$

d) $\frac{x^2 - 4x + 3}{x^2 + 2x - 15}$

Solution: We factor both numerator and denominator and then simplify. We can easily factor both of these polynomials by trial and error.

$$\frac{x^2 - 4x + 3}{x^2 + 2x - 15} = \frac{(x - 3)(x - 1)}{(x + 5)(x - 3)} = \frac{\cancel{(x - 3)}(x - 1)}{(x + 5)\cancel{(x - 3)}} = \boxed{\frac{x - 1}{x + 5}}$$

e) $\frac{(x + 5) - 2}{5(x + 2) - (x - 2)}$

Solution: We simplify both numerator and denominator, then if possible, factor these and then simplify the fraction by cancellation.

$$\frac{(x + 5) - 2}{5(x + 2) - (x - 2)} = \frac{x + 5 - 2}{5x + 10 - x + 2} = \frac{x + 3}{4x + 12} = \frac{x + 3}{4(x + 3)} = \frac{\cancel{x + 3}}{4\cancel{(x + 3)}} = \boxed{\frac{1}{4}}$$

2. Perform the indicated operations and simplify.

a) $\frac{c}{5a} \cdot \frac{15a^2b}{3b^2c}$

Solution: We perform the multiplication among fractions (top by top, bottom by bottom) and then simplify by canceling factors appearing in both the numerator and denominator.

$$\frac{c}{5a} \cdot \frac{15a^2b}{3b^2c} = \frac{15a^2bc}{15ab^2c} = \frac{15abc}{15abc} \cdot \frac{a}{b} = \frac{15\cancel{abc}a}{15\cancel{abc}b} = \boxed{\frac{a}{b}}$$

b) $\frac{5x-30}{x^2-36} \cdot \frac{3x+18}{5}$

Solution: we will factor whatever we can and then simplify by canceling factors appearing in both the numerator and denominator.

$$\frac{5x-30}{x^2-36} \cdot \frac{3x+18}{5} = \frac{5(x-6)}{(x+6)(x-6)} \cdot \frac{3(x+6)}{5} = \frac{\cancel{5}(x-\cancel{6})}{(x+6)(\cancel{x-6})} \cdot \frac{3(x+6)}{\cancel{5}} = \frac{1}{x+6} \cdot \frac{3(x+\cancel{6})}{1} = \boxed{3}$$

c) $\frac{x^2-3x}{x^2-8x+15} \cdot \frac{x^2-16x+15}{x^2-x}$

Solution: We factor whatever we can and then simplify by canceling factors appearing in both the numerator and denominator. We can factor all of these polynomials by completing the square or by factoring out the greatest common factor.

$$\begin{aligned} \frac{x^2-3x}{x^2-8x+15} \cdot \frac{x^2-16x+15}{x^2-x} &= \frac{x(x-3)}{(x-3)(x-5)} \cdot \frac{(x-1)(x-15)}{x(x-1)} = \frac{x(\cancel{x-3})}{(\cancel{x-3})(x-5)} \cdot \frac{(\cancel{x-1})(x-15)}{x(\cancel{x-1})} \\ &= \frac{\cancel{x}}{x-5} \cdot \frac{x-15}{\cancel{x}} = \boxed{\frac{x-15}{x-5}} \end{aligned}$$

d) $\frac{x^2-9}{x^2-4x-21} \div \frac{4x-12}{3x-21}$

Solution: We first re-write the division as multiplication by the reciprocal.

$$\frac{x^2-9}{x^2-4x-21} \div \frac{4x-12}{3x-21} = \frac{x^2-9}{x^2-4x-21} \cdot \frac{3x-21}{4x-12}$$

We now factor the polynomials appearing in the fractions

$$\begin{aligned} x^2-9 &= (x+3)(x-3) & 4x-12 &= 4(x-3) \\ x^2-4x-21 &= (x+3)(x-7) & 3x-21 &= 3(x-7) \end{aligned}$$

We now re-write the fractions using these factored forms, and cancel out factors appearing in both numerator and denominator of the product.

$$\begin{aligned} \frac{x^2-9}{x^2-4x-21} \cdot \frac{3x-21}{4x-12} &= \frac{(x+3)(x-3)}{(x+3)(x-7)} \cdot \frac{3(x-7)}{4(x-3)} = \frac{(\cancel{x+3})(x-3)}{(\cancel{x+3})(x-7)} \cdot \frac{3(x-7)}{4(x-3)} = \frac{3(x-3)(x-7)}{4(x-3)(x-7)} \\ &= \frac{3(\cancel{x-3})(\cancel{x-7})}{4(\cancel{x-3})(\cancel{x-7})} = \boxed{\frac{3}{4}} \end{aligned}$$

$$e) \frac{x^2 - 10x + 25}{x^2 - 10x + 24} \left(\frac{x^2 - 2x - 8}{x^2 - 6x + 5} \div \frac{x - 5}{x - 1} \right)$$

Solution: We first re-write the division as multiplication by the reciprocal.

$$\frac{x^2 - 10x + 25}{x^2 - 10x + 24} \left(\frac{x^2 - 2x - 8}{x^2 - 6x + 5} \div \frac{x - 5}{x - 1} \right) = \frac{x^2 - 10x + 25}{x^2 - 10x + 24} \left(\frac{x^2 - 2x - 8}{x^2 - 6x + 5} \cdot \frac{x - 1}{x - 5} \right)$$

We now factor the polynomials appearing in each fraction.

$$\begin{aligned} x^2 - 10x + 25 &= (x - 5)^2 & x^2 - 2x - 8 &= (x + 2)(x - 4) \\ x^2 - 10x + 24 &= (x - 4)(x - 6) & x^2 - 6x + 5 &= (x - 1)(x - 5) \end{aligned}$$

We now re-write the problem, using the factored form of polynomials.

$$\frac{x^2 - 10x + 25}{x^2 - 10x + 24} \left(\frac{x^2 - 2x - 8}{x^2 - 6x + 5} \cdot \frac{x - 1}{x - 5} \right) = \frac{(x - 5)^2}{(x - 4)(x - 6)} \left(\frac{(x + 2)(x - 4)}{(x - 1)(x - 5)} \cdot \frac{x - 1}{x - 5} \right)$$

We now perform the multiplication within the parentheses. Notice that we can cancel out $x - 1$.

$$\begin{aligned} &= \frac{(x - 5)^2}{(x - 4)(x - 6)} \left(\frac{(x + 2)(x - 4)}{(\cancel{x - 1})(x - 5)} \cdot \frac{\cancel{x - 1}}{x - 5} \right) = \frac{(x - 5)^2}{(x - 4)(x - 6)} \cdot \frac{(x + 2)(x - 4)}{(x - 5)^2} \\ &= \frac{(\cancel{x - 5})^2}{(x - 4)(x - 6)} \cdot \frac{(x + 2)(x - 4)}{(\cancel{x - 5})^2} = \frac{1}{(\cancel{x - 4})(x - 6)} \cdot \frac{(x + 2)(\cancel{x - 4})}{1} \\ &= \boxed{\frac{x + 2}{x - 6}} \end{aligned}$$

f) Recall that to divide is to multiply by the reciprocal.

$$\frac{6x^2y^2 - 36x^2y + 48x^2}{py^2 - 2py - 8p} \div \frac{-3y^2 + 24y - 36}{p^2y + 2p^2} = \frac{6x^2y^2 - 36x^2y + 48x^2}{py^2 - 2py - 8p} \cdot \frac{p^2y + 2p^2}{-3y^2 + 24y - 36}$$

As before, we start with four factoring exercises. As always, we start with the GCF.

$$6x^2y^2 - 36x^2y + 48x^2 = 6x^2(y^2 - 6y + 8) = 6x^2(y - 2)(y - 4)$$

$$py^2 - 2py - 8p = p(y^2 - 2y - 8) = p(y + 2)(y - 4)$$

$$p^2y + 2p^2 = p^2(y + 2)$$

$$-3y^2 + 24y - 36 = -3(y^2 - 8y + 12) = -3(y - 2)(y - 6)$$

$$\frac{6x^2y^2 - 36x^2y + 48x^2}{py^2 - 2py - 8p} \cdot \frac{p^2y + 2p^2}{-3y^2 + 24y - 36} = \frac{6x^2(y - 2)(y - 4)}{p(y + 2)(y - 4)} \cdot \frac{p^2(y + 2)}{-3(y - 2)(y - 6)}$$

$$= \frac{6x^2(y - 2)(\cancel{y - 4})}{p(y + 2)(\cancel{y - 4})} \cdot \frac{p^2(y + 2)}{-3(y - 2)(y - 6)} = \frac{6x^2(y - 2)}{p(\cancel{y + 2})} \cdot \frac{p^2(\cancel{y + 2})}{-3(y - 2)(y - 6)}$$

$$= \frac{6x^2(\cancel{y - 2})}{\cancel{p}} \cdot \frac{\cancel{p} \cdot p}{-3(\cancel{y - 2})(y - 6)} = \frac{2 \cdot 3x^2}{1} \cdot \frac{p}{-3(y - 6)} = \frac{2x^2}{1} \cdot \frac{p}{-(y - 6)} = \boxed{\frac{-2px^2}{y - 6}}$$

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