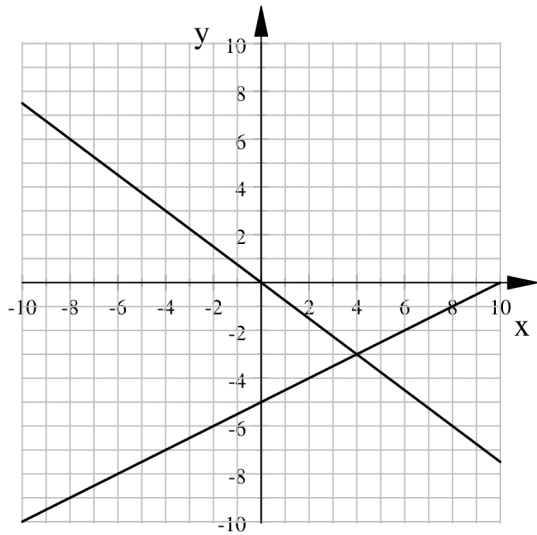


Recall that the graph of an equation is the set of all points whose coordinates form a solution to the equation. Consider the lines $y = -\frac{3}{4}x$ and $x - 2y = 10$. We graph these lines in the same coordinate system. What can we say about the point where the two lines intersect each other? In this particular case, we can read the coordinates of this point: $(4, -3)$.



This point is on the line $y = -\frac{3}{4}x$ if and only if its coordinates form a solution of the equation. We check:

$$\text{LHS} = -3 \quad \text{and} \quad \text{RHS} = -\frac{3}{4}(4) = -3 \checkmark$$

Therefore, this point is on the line $y = -\frac{3}{4}x$.

Similarly, this point is on the line $x - 2y = 10$ if and only if its coordinates form a solution of the equation.

$$\text{LHS} = 4 - 2(-3) = 4 + 6 = 10 \quad \text{and} \quad \text{RHS} = 10 \checkmark$$

Therefore, this point is also on the line $x - 2y = 10$.

Two different lines cannot have more than one point in common. Two points uniquely determine a line. Therefore, if two lines have two points in common, they are really the same line. In this sense, the intersection point is special. Algebraically, the intersection point is the unique point whose coordinates are a solution for the equations of both lines.

Definition: Two equations in x and y form a **system of equations**. The **solution(s)** of the system are the point(s) whose coordinates form a solution of both equations. To **solve** a system means to find all solutions of it.

In our example above, $(-4, 3)$ is the only solution of the system $\begin{cases} y = -\frac{3}{4}x \\ x - 2y = 10 \end{cases}$.

In this case, the intersection was a point whose both coordinates happen to be integers. (Such points are called lattice points.) In case we are less lucky, we will need more precise tools for solving than graphing the two equations in the same coordinate system.

There are several algebraic methods to solve a system of linear equations.

Solving Linear Systems Using Elimination.

This method is sometimes called the addition method. We will refer to it as elimination. The basic principle behind this method is the following. We can multiply both sides of an equation by the same non-zero number- and this step will not change the solution set. We can also add equations: If $x = y$ and $A = B$, then $x + A = y + B$.

Example 1. Solve the given system of linear equations using elimination.
$$\begin{cases} 3x - 5y = 11 \\ 2x + 3y = 20 \end{cases}$$

Solution: We will multiply both sides of one or both equations with the goal to end up with one unknown with opposite coefficients. This is possible for both x and y . To eliminate y , we will multiply the first equation by 3 and the second equation by 5. That way the coefficients of y will be 15 and -15 . Once we add the two equations, y will be eliminated.

$$\begin{cases} 3x - 5y = 11 & \text{multiply by 3} \\ 2x + 3y = 20 & \text{multiply by 5} \end{cases} \implies \begin{cases} 9x - 15y = 33 \\ 10x + 15y = 100 \end{cases}$$

We will eliminate y by adding the two equations (i.e adding the left-hand side to left-hand side and right-hand side to right-hand side.) Then the equation will become an equation in only x , so we can solve for it.

$$\begin{array}{r} \begin{cases} 9x - 15y = 33 \\ + \quad 10x + 15y = 100 \end{cases} \\ \hline 19x = 133 \quad \text{divide by 19} \\ x = 7 \end{array}$$

Now that we know the value of x , we can use either one of the two equations to find the value of y . The second equation, in its original form, will be transformed from $2x + 3y = 20$ to $2 \cdot 7 + 3y = 20$. Now we can easily solve for y .

$$\begin{array}{r} 2 \cdot 7 + 3y = 20 \\ 14 + 3y = 20 \quad \text{subtract} \\ 3y = 6 \\ y = 2 \end{array}$$

Thus, the solution of this system is $x = 7$ and $y = 2$, or, in short, $(7, 2)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $3x - 5y = 11$	Checking $2x + 3y = 20$
LHS = $3 \cdot 7 - 5 \cdot 2 = 21 - 10 = 11$	LHS = $2 \cdot 7 + 3 \cdot 2 = 14 + 6 = 20$
RHS = $11 \checkmark$	RHS = $20 \checkmark$

Therefore, our solution, $\boxed{(7, 2)}$ is correct.

Example 2. Solve the given system of linear equations using elimination.
$$\begin{cases} 2x - y = -19 \\ -x + 3y = 12 \end{cases}$$

Solution: We will multiply both sides of one or both equations with the goal to end up with one unknown with opposite coefficients. This is possible for both x and y . To eliminate x , we will leave the first equation as it is, and multiply the second equation by 2. This way the coefficients of x will be 2 and 2. Then, when we add the two equations, x will be eliminated, and we can solve the equation for y .

$$\begin{cases} 2x - y = -19 \\ -x + 3y = 12 & \text{multiply by 2} \end{cases} \implies \begin{cases} 2x - y = -19 \\ -2x + 6y = 24 \end{cases}$$

We now add the two equations:

$$\begin{array}{r} \left\{ \begin{array}{l} 2x - y = -19 \\ + \quad -2x + 6y = 24 \end{array} \right. \\ \hline 5y = 5 \quad \text{divide by 5} \\ y = 1 \end{array}$$

Now that we know the value of y , we can use either one of the two equations to find the value of x . The first equation, in its original form, will be transformed from $2x - y = -19$ to $2x - 1 = -19$. Now we can easily solve for x .

$$\begin{array}{r} 2x - 1 = -19 \quad \text{add 1} \\ 2x = -18 \quad \text{divide by 2} \\ x = -9 \end{array}$$

Thus, the solution of this system is $x = -9$ and $y = 1$, or, in short, $(-9, 1)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $2x - y = -19$ LHS = $2(-9) - 1 = -18 - 1 = -19$ RHS = $-19 \checkmark$	Checking $-x + 3y = 12$ LHS = $-(-9) + 3 \cdot 1 = 9 + 3 = 12$ RHS = $12 \checkmark$
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Therefore, our solution, $\boxed{(-9, 1)}$ is correct.

Most real-world problems boil down to systems of equations. In this sense, solving systems of equations is one of the most important tasks in problem solving.

Example 3. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens and how many cows are there on the farm?

Solution: We will denote the number of chickens by x and the number of cows by y . The first equation will express the number of heads. x many chickens come with x many heads, and y many cows come with y many heads. The second equation will express the number of legs. x many chickens come with $2x$ many legs, and y many cows come with $4y$ many heads.

$$\begin{cases} x + y = 53 \\ 2x + 4y = 174 \end{cases}$$

Before we start solving the system, let us notice that we can simplify the second equation by dividing both sides by 2. We can often make our life easier with simplifications such as this one.

$$\begin{cases} x + y = 53 \\ x + 2y = 87 \end{cases}$$

To eliminate x , we will multiply the first equation by -1 leave the second equation as is. Then we add the two equations.

$$\begin{array}{r} \left\{ \begin{array}{l} -x - y = -53 \\ + \quad x + 2y = 87 \end{array} \right. \\ \hline \end{array}$$

$$y = 34$$

Now that we know the value of y , we use the first equation to find x .

$$\begin{aligned} x + 34 &= 53 && \text{subtract 34} \\ x &= 19 && \implies x = 19, y = 34 \end{aligned}$$

Thus we have 19 chickens and 34 cows. We check: the number of heads is $19 + 34 = 53$, and the number of legs is $2 \cdot 19 + 4 \cdot 34 = 38 + 136 = 174$. So our solution is correct.

Example 4. We invested \$10 000 into two bank accounts. One account earns 14% per year, the other account earns 8% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1238?

Solution: Let us denote the amount invested at 14% by x and the amount invested at 8% by y . The two equations will express the total amount invested, and the total interest earned.

$$\begin{aligned} x + y &= 10\,000 && \text{the amounts invested add up to \$10 000} \\ 0.14x + 0.08y &= 1238 && \text{the interests earned add up to \$1238} \end{aligned}$$

We solve the system of equation by elimination. But let us first make the second equation simpler:

$$\begin{aligned} 0.14x + 0.08y &= 1238 && \text{multiply by 100} \\ 14x + 8y &= 123\,800 && \text{divide by 2} \\ 7x + 4y &= 61\,900 \end{aligned}$$

We now have

$$\begin{aligned} x + y &= 10\,000 \\ 7x + 4y &= 61\,900 \end{aligned}$$

We will multiply the first equation by -4 to eliminate y . Then we add the two equations.

$$\begin{array}{rcl} -4x - 4y &= & -40\,000 \\ 7x + 4y &= & 61\,900 \\ \hline 3x &= & 21\,900 && \text{divide by 3} \\ x &= & 7300 \end{array}$$

Thus we invested \$7300 at 14%. The other amount can be found using the first equation:

$$\begin{aligned} 7300 + y &= 10\,000 \\ y &= 2700 \end{aligned}$$

We invested \$7300 at 14% and \$2700 at 8%. We check: the amounts add up to $7300 + 2700 = 10\,000$. The interest from the accounts are:

$$14\% \text{ of } 7300 \text{ is } 0.14(7300) = 1022 \text{ and } 8\% \text{ of } 2700 \text{ is } 0.08(2700) = 216$$

Since $1022 + 216 = 1238$, our solution is correct.

Example 5. We have a jar of coins, all pennies and dimes. All together, we have 372 coins, and the total value of all coins in the jar is \$20.91. How many pennies are there in the jar?

Solution: Let us denote the number of pennies by x and the number of dimes by y . The first equation will express the number of the coins. This equation is therefore $x + y = 372$. To express the value of all coins, x many pennies are worth $0.01x$ and y many dimes are worth $0.1y$. The total value of all coins is then

$$0.01x + 0.1y = 20.91$$

In order to clear the decimals, we may multiply both sides by 100. Then we have

$$x + 10y = 2091$$

Let us notice that this is the same equation that we would obtain if we expressed the value of all coins in cents and not in dollars. So our system is now

$$\begin{array}{rcl} x + y & = & 372 \\ x + 10y & = & 2091 \end{array}$$

We will eliminate x by multiplying the first equation by -1 and leaving the second equation as is. Then we add the two equations and solve for y .

$$\begin{array}{rcl} -x - y & = & -372 \\ x + 10y & = & 2091 \\ \hline 9y & = & 1719 \quad \text{divide by 9} \\ y & = & 191 \end{array}$$

Thus we have 191 dimes. The number of pennies can be found using any of the two equations. We will use the simplest one, the original first equation.

$$\begin{array}{rcl} x + 191 & = & 372 \quad \text{subtract 191} \\ x & = & 181 \end{array}$$

Thus we have 181 pennies and 191 dimes. We check: the number of all coins is $181 + 191 = 372$, and the value of the coins is $0.01 \cdot 181 + 0.1 \cdot 191 = 1.81 + 19.1 = 20.91$. Thus our solution is correct.



Practice Problems

1. Solve each of the following system of linear equations.

$$a) \begin{cases} 2x - y = -8 \\ x + 2y = -9 \end{cases}$$

$$d) \begin{cases} \frac{1}{2}x + \frac{1}{4}y = -1 \\ \frac{1}{2}y - \frac{1}{3}x = 6 \end{cases}$$

$$g) \begin{cases} 3x - 2y = 2 \\ 2x + 3y = 5 \end{cases}$$

$$b) \begin{cases} 2(p - 1) - 3(q - 1) = 24 \\ p + q = -6 \end{cases}$$

$$e) \begin{cases} 3x - y = 4 \\ 2(y - 3) = -2(x + 1) \end{cases}$$

$$h) \begin{cases} x - 2y = 5 \\ 2x + 3y = 10 \end{cases}$$

$$c) \begin{cases} 3x - y = 10 \\ \frac{1}{3}x - y = 2 \end{cases}$$

$$f) \begin{cases} 2a + b = 17 \\ a + b = 5 \end{cases}$$

$$i) \begin{cases} 0.5x - 1.2y = -1.21 \\ x + 3.2y = 2.06 \end{cases}$$

2. Given the equations of two straight lines, find both coordinates of all intersection points.

$$a) 2x - 5y = -41 \text{ and } x + y = 4$$

$$d) 3x - y = 9 \text{ and } -\frac{2}{3}x + \frac{1}{2}y = -2$$

3. There is an animal farm where chickens and cows live. All together, there are 60 heads and 164 legs. How many chickens and how many cows are there on the farm?
4. We invested \$6000 into two bank accounts. One account earns 7% per year, the other account earns 11% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$520?
5. We have 51 coins, all dimes and quarters, in the total value of \$7.05. How many quarters and how many dimes are there?
6. We invested \$7600 in two bank accounts. One account earns 9% per year, the other account earns 13% per year. How much did we invest into each account if after the first year we have a total of \$8508 in the accounts?



Answers

Practice Problems

1. a) $x = -5, y = -2$ b) $p = 1, q = -7$ c) $x = 3, y = -1$ d) $x = -6, y = 8$
 e) $x = \frac{3}{2}, y = \frac{1}{2}$ f) $a = 12, b = -7$ g) $x = \frac{16}{13}, y = \frac{11}{13}$ h) $x = 5, y = 0$
 i) $x = -0.5, y = 0.8$
2. a) $(-3, 7)$ b) $(3, 0)$ 3. 38 chickens, 22 cows 4. \$3500 at 7% and \$2500 at 11%
5. 38 dimes and 13 quarters 6. \$2000 at 9% and \$5600 at 13%

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