

Rational expressions are fractions in which at least one of numerator and denominator is an algebraic expression. For example, $\frac{3x-1}{x^2+1}$, $\frac{a+b}{a^2-3b}$, and $\frac{m}{m+1}$ are rational expressions. In what follows, we will see how rational expressions can be simplified, multiplied and divided, just like the old, familiar fractions formed from integers.

We can simplify algebraic expressions by dividing both numerator and denominator by the same quantity. This process will be similar to reducing fractions to lowest terms. For example, we can reduce the fraction $\frac{12}{20}$ by dividing both numerator and denominator by 4.

$$\frac{12}{20} = \frac{4 \cdot 3}{4 \cdot 5} = \frac{3}{5}$$

Note that dividing both numerator and denominator is the same as removing the same factor from their factored form. In case of a rational expression, we will factor both numerator and denominator and if there is a common factor, we remove it from both.

Example 1. Simplify the rational expression $\frac{x^3 - x}{x + 1}$.

Solution: In this case we only factor the numerator since the denominator can not be factored. After we factor out the greatest common factor (or GCF) which is x , the expression can be factored via the difference of squares theorem.

$$x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1)$$

Then we simplify the fraction by canceling out the same factor from numerator and denominator.

$$\frac{x^3 - x}{x + 1} = \frac{x(x + 1)(x - 1)}{x + 1} = \boxed{x(x - 1) \text{ or } x^2 - x}$$

Example 2. Simplify the rational expression $\frac{x^2 - 2x - 35}{x^2 + 2x - 15}$.

Solution: We factor both numerator and denominator and then simplify by cancelling out common factors. We can easily factor both of these polynomials by trial and error or by completing the square.

$$\begin{array}{ll} N &= x^2 - 2x - 35 \\ &= x^2 - 2x + 1 - 1 - 35 \\ &= (x - 1)^2 - 36 \\ &= (x - 1)^2 - 6^2 \\ &= (x - 1 + 6)(x - 1 - 6) \\ &= (x + 5)(x - 7) \\ D &= x^2 + 2x - 15 \\ &= x^2 + 2x + 1 - 1 - 15 \\ &= (x + 1)^2 - 16 \\ &= (x + 1)^2 - 4^2 \\ &= (x + 1 + 4)(x + 1 - 4) \\ &= (x + 5)(x - 3) \end{array}$$

We can now simplify the expression by cancelling out the common factor, $x + 5$.

$$\frac{x^2 - 2x - 35}{x^2 + 2x - 15} = \frac{(x + 5)(x - 7)}{(x + 5)(x - 3)} = \boxed{\frac{x - 7}{x - 3}}$$

The following example will illustrate that sometimes we can simplify a rational expression even if neither numerator nor denominator can be factored.

Example 3. Simplify the rational expression $\frac{2a-5}{5-2a}$.

Solution: We need to notice that the numerator and denominator are opposites of each other. Indeed, the opposite of $2a-5$ is $5-2a$ since $-1(2a-5) = -2a+5 = 5-2a$. Therefore, we have a fraction such as $\frac{3}{-3}$ or $\frac{-10}{10}$. We factor out -1 from either the numerator or the denominator (we selected the denominator) and then cancel out the common factor.

$$\frac{2a-5}{5-2a} = \frac{2a-5}{-1(2a-5)} = \frac{1(2a-5)}{-1(2a-5)} = \frac{1}{-1} = \boxed{-1}$$

We can multiply and divide rational expressions just like fractions.

Example 4. Perform the indicated operations and simplify.

$$\text{a) } \frac{5x-30}{x^2-36} \cdot \frac{3x+18}{5} \quad \text{b) } \frac{x^2-3x}{x^2-8x+15} \cdot \frac{x^2-16x+15}{x^2-x}$$

Solution: a) we will factor each expression (if we can) and then simplify by cancelling factors appearing in both the numerator and denominator. Just like in multiplying fractions, we can cancel out the same factor from the numerator of one fraction and the denominator of the other.

$$5x-30 = 5(x-6), \quad x^2-36 = (x+6)(x-6), \quad \text{and} \quad 3x+18 = 3(x+6).$$

$$\frac{5x-30}{x^2-36} \cdot \frac{3x+18}{5} = \frac{5(x-6)}{(x+6)(x-6)} \cdot \frac{3(x+6)}{5} = 3$$

b) We factor whatever we can and then simplify by canceling factors appearing in both the numerator and denominator. We can factor all of these polynomials by completing the square or by factoring out the greatest common factor.

$$\text{First expression: } x^2-3x = x(x-3) \quad \text{and} \quad x^2-8x+15 = (x-3)(x-5)$$

$$\text{Second expression: } x^2-16x+15 = (x-1)(x-15) \quad \text{and} \quad x^2-x = x(x-1)$$

$$\frac{x^2-3x}{x^2-8x+15} \cdot \frac{x^2-16x+15}{x^2-x} = \frac{x(x-3)}{(x-3)(x-5)} \cdot \frac{(x-1)(x-15)}{x(x-1)} = \frac{x}{x-5} \cdot \frac{x-15}{x} = \boxed{\frac{x-15}{x-5}}$$

Example 5. Perform the indicated operations and simplify. $\frac{x^2-9}{x^2-4x-21} \div \frac{4x-12}{3x-21}$

Solution: We first re-write the division as multiplication by the reciprocal.

$$\frac{x^2-9}{x^2-4x-21} \div \frac{4x-12}{3x-21} = \frac{x^2-9}{x^2-4x-21} \cdot \frac{3x-21}{4x-12}$$

We now factor the polynomials appearing in the fractions

$$\begin{array}{ll} x^2-9 &= (x+3)(x-3) & 4x-12 &= 4(x-3) \\ x^2-4x-21 &= (x+3)(x-7) & 3x-21 &= 3(x-7) \end{array}$$

We re-write the fractions using these factored forms, and cancel out factors appearing in both numerator and denominator of the product.

$$\frac{x^2-9}{x^2-4x-21} \cdot \frac{3x-21}{4x-12} = \frac{(x+3)(x-3)}{(x+3)(x-7)} \cdot \frac{3(x-7)}{4(x-3)} = \frac{3(x+3)(x-3)(x-7)}{4(x+3)(x-3)(x-7)} = \frac{3}{4}$$



Practice Problems

1. Simplify each of the following.

a) $\frac{2b-5}{10-4b}$ c) $\frac{4t^2-9}{4t-6}$ e) $\frac{6m+m^2-216}{6m-m^2+72}$ g) $\frac{-7x-11-3(x-2)}{5x-11+3(x+5)}$

b) $\frac{x^2-1}{x+1}$ d) $\frac{p^2-p}{p^2-1}$ f) $\frac{x^2-144x-900}{x^2-158x+1200}$

2. Perform the indicated operations and simplify.

a) $\frac{x}{5yz} \cdot \frac{10x^2y^3z}{4xy^2}$ c) $\frac{x^2-5x}{x^2-2x-15} \cdot \frac{x^2-9}{x^2-3x}$ e) $\frac{5y-35}{y^2-2y-35} \cdot \frac{3y+15}{5y-5}$

b) $\frac{a^2-8a+16}{a} \cdot \frac{a^3}{4-a}$ d) $\frac{x^2-4x-21}{x^2-49} \div \frac{8x+x^2+15}{2x+x^2-35}$ f) $\frac{2x^2-98}{x^2-6x-7} \div \frac{21+3x}{6x^2-6}$



Answers

1. a) $-\frac{1}{2}$ b) $x - 1$ c) $\frac{2t + 3}{2}$ d) $\frac{p}{p + 1}$ e) $-\frac{m + 18}{m + 6}$ f) $\frac{x + 6}{x - 8}$ g) $-\frac{5}{4}$

2. a) $\frac{x^2}{2}$ b) $-a^2(a - 4)$ c) 1 d) $\frac{x - 5}{x + 5}$ e) $\frac{3}{y - 1}$ f) $4x - 4$