

We have been studying fractions formed from polynomials. Such abstract fractions are called rational expressions. In case of the usual fractions multiplication and division are simpler to perform than are addition and subtraction. This is because when we add or subtract fractions, we must bring them to the common denominator. This will also be the case in rational expressions.

One difference is that when we subtract algebraic expressions, we need to make sure not to violate the distributive law.

Example 1. Perform the indicated operations.

$$\text{a) } \frac{5}{6} - \frac{1}{4} \qquad \text{b) } \frac{2x-1}{10} - \frac{x-3}{8}$$

Solution: a) We need to bring the fractions to the common denominator. Recall that we may multiply both numerator and denominator of fractions without changing their values. The least common multiple of 4 and 6 is 12.

To re-write $\frac{5}{6}$ with a denominator of 12, we multiply both numerator and denominator by 2. To re-write $\frac{1}{4}$ with a denominator of 12, we multiply both numerator and denominator by 3. Once they have the same denominator, we subtract the numerators.

$$\frac{5}{6} - \frac{1}{4} = \frac{5 \cdot 2}{6 \cdot 2} - \frac{1 \cdot 3}{4 \cdot 3} = \frac{10}{12} - \frac{3}{12} = \frac{10-3}{12} = \boxed{\frac{7}{12}}$$

b) The computation with rational expressions is similar. The only difference is between subtracting numbers and subtracting algebraic expressions. We need to watch out for the distributive law.

The least common multiple of 10 and 8 is 40. To re-write $\frac{2x-1}{10}$ with a denominator of 40, we multiply both numerator and denominator by 4. To re-write $\frac{x-3}{8}$ with a denominator of 40, we multiply both numerator and denominator by 5. Once they have the same denominator, we subtract the numerators. This time however, the numerators are algebraic expressions.

$$\begin{aligned} \frac{2x-1}{10} - \frac{x-3}{8} &= \frac{(2x-1) \cdot 4}{10 \cdot 4} - \frac{(x-3) \cdot 5}{8 \cdot 5} = \frac{4(2x-1)}{40} - \frac{5(x-3)}{40} \\ &= \frac{4(2x-1) - 5(x-3)}{40} = \frac{8x-4-5x+15}{40} = \boxed{\frac{3x+11}{40}} \end{aligned}$$

It is good strategy to wait with distributing 4 and 5 in the numerators. Consider the following computations:

$$\frac{4(2x-1)}{40} - \frac{5(x-3)}{40} = \frac{8x-4}{40} - \frac{5x-15}{40} = \frac{8x-4-5x \boxed{-} 15}{40} \text{ is incorrect}$$

$$\frac{4(2x-1)}{40} - \frac{5(x-3)}{40} = \frac{8x-4}{40} - \frac{5x-15}{40} = \frac{8x-4-(5x-15)}{40} \text{ is correct}$$

$$\frac{4(2x-1)}{40} - \frac{5(x-3)}{40} = \frac{4(2x-1) - 5(x-3)}{40} = \frac{8x-4-5x+15}{40} \text{ is safer}$$

Example 2. Perform the indicated operations.

$$\text{a) } \frac{7x-1}{x^2-1} - \frac{4}{x+1} \quad \text{b) } \frac{6}{2x+x^2} + \frac{1-x}{x+2}$$

Solution: a) In case of numbers, we do not have to find the least common denominator, any common multiple will get us to the correct answer. However, in case of rational expressions we need to find the least common multiple. Otherwise, we might end up with polynomials of degree too high for us to factor. To find the least common multiple, we need to factor first the denominators.

$$\frac{7x-1}{x^2-1} - \frac{4}{x+1} = \frac{7x-1}{(x+1)(x-1)} - \frac{4}{x+1}$$

The common denominator is $(x+1)(x-1)$. To bring $\frac{4}{x+1}$ to the common denominator, we multiply both numerator and denominator by $x-1$.

$$\begin{aligned} \frac{7x-1}{x^2-1} - \frac{4}{x+1} &= \frac{7x-1}{(x+1)(x-1)} - \frac{4}{x+1} = \frac{7x-1}{(x+1)(x-1)} - \frac{4(x-1)}{(x+1)(x-1)} = \frac{7x-1-4(x-1)}{(x+1)(x-1)} \\ &= \frac{7x-1-4x+4}{(x+1)(x-1)} = \frac{3x+3}{(x+1)(x-1)} \end{aligned}$$

Just as with fractions, we need to reduce the result to lowest terms. To do that, we factor the numerator and check for common factors to cancel out.

$$\frac{3x+3}{(x+1)(x-1)} = \frac{3(x+1)}{(x+1)(x-1)} = \frac{\cancel{3(x+1)}}{\cancel{(x+1)}(x-1)} = \boxed{\frac{3}{x-1}}$$

b) The first denominator is $x^2+2x = x(x+2)$. The other denominator is $x+2$. Therefore, the common denominator is $x(x+2)$.

$$\begin{aligned} \frac{6}{2x+x^2} + \frac{1-x}{x+2} &= \frac{6}{x(x+2)} + \frac{(1-x) \cdot x}{(x+2) \cdot x} = \frac{6}{x(x+2)} + \frac{x(1-x)}{x(x+2)} = \frac{6+x(1-x)}{x(x+2)} \\ &= \frac{6+x(1-x)}{x(x+2)} = \frac{6+x-x^2}{x(x+2)} = \frac{-x^2+x+6}{x(x+2)} \end{aligned}$$

Before we present the final answer, we need to make sure it is reduced to lowest terms. So we need to factor the numerator, $-x^2+x+6$.

$$-x^2+x+6 = -(x^2-x-6) = -(x-3)(x+2)$$

$$\text{So our final answer is } \frac{-x^2+x+6}{x(x+2)} = \frac{-(x-3)(x+2)}{x(x+2)} = \frac{-(x-3)\cancel{(x+2)}}{x\cancel{(x+2)}} = \boxed{\frac{-x+3}{x}}$$



Sample Problems

Perform the indicated operations and simplify.

$$1. \frac{3x}{x-2} - \frac{x+4}{x-2}$$

$$2. \frac{10}{x-y} - \frac{5}{y-x}$$

$$3. \frac{1}{x-y} - \frac{1}{x+y}$$

$$4. \frac{2}{p-5} - \frac{p+11}{p^2-2p-15}$$

$$5. \frac{x^2-5x+78}{18x+x^2-208} - \frac{x}{x+26}$$

$$6. \left[\left(\frac{1}{b} - b \right) \div \left(1 - \frac{1}{b} \right) \right] (1+b)$$



Practice Problems

Perform the indicated operations and simplify.

$$1. \frac{x+y}{2} - \frac{x-y}{2}$$

$$2. \frac{3a}{a-1} - \frac{a+2}{a-1}$$

$$3. \frac{a+1}{a-b} - \frac{b-1}{b-a}$$

$$4. \frac{5y+1}{y+5} + \frac{y^2-1}{y+5}$$

$$5. \frac{1}{p-5} - \frac{1}{p+5}$$

$$6. \frac{x-5}{x+2} - \frac{3}{2-x} - \frac{14-x}{x^2-4}$$

$$7. \frac{3}{x+1} - \frac{2}{x-1} + \frac{x+3}{x^2-1}$$

$$8. \frac{2(m+2)}{4m+m^2-12} - \frac{1}{m-2}$$

$$9. \frac{1}{x-2} - \frac{3}{x+6}$$



Answers

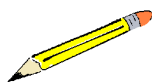
Sample Problems

1. 2 2. $\frac{15}{x-y}$ or $-\frac{15}{y-x}$ 3. $\frac{2y}{(x-y)(x+y)}$ or $\frac{2y}{x^2-y^2}$ 4. $\frac{1}{p+3}$
5. $\frac{3}{x-8}$ 6. $-(b+1)^2$

Practice Problems

1. y 2. 2 3. $\frac{a+b}{a-b}$ 4. y 5. $\frac{10}{p^2-25}$ 6. $\frac{x-1}{x+2}$
7. $\frac{2}{x+1}$ 8. $\frac{1}{m+6}$ 9. $\frac{-2x+12}{x^2+4x-12}$

Sample Problems



Solutions

Perform the indicated operations and simplify.

1. a) $\frac{3x}{x-2} - \frac{x+4}{x-2}$

Solution: This is a subtraction of fractions. The denominators are the same, the only difficulty is that we are subtracting expressions instead of numbers. The second pair of parentheses is essential.

$$\frac{3x}{x-2} - \frac{x+4}{x-2} = \frac{(3x) - (x+4)}{x-2} = \frac{3x - x - 4}{x-2} = \frac{2x - 4}{x-2} = \frac{2(x-2)}{x-2} = \frac{\cancel{2(x-2)}}{\cancel{(x-2)}} = \boxed{2}$$

2. $\frac{10}{x-y} - \frac{5}{y-x}$

Solution: This problem is much easier when we realize that the denominators are opposites of each other, since the opposite of $y-x$ is $-1(y-x) = -y+x = x-y$

$$\frac{10}{x-y} - \frac{5}{y-x} = \frac{10}{x-y} - \frac{-1 \cdot 5}{-1 \cdot (y-x)} = \frac{10}{x-y} - \frac{-5}{x-y} = \frac{10 - (-5)}{x-y} = \boxed{\frac{15}{x-y}}$$

3. $\frac{1}{x-y} - \frac{1}{x+y}$

Solution: We first bring the two fractions to the common denominator, which is $(x+y)(x-y) = x^2 - y^2$

$$\begin{aligned} \frac{1}{x-y} - \frac{1}{x+y} &= \frac{1 \cdot (x+y)}{(x-y)(x+y)} - \frac{1 \cdot (x-y)}{(x+y)(x-y)} \\ &= \frac{x+y}{(x-y)(x+y)} - \frac{x-y}{(x-y)(x+y)} \end{aligned}$$

We are now ready to subtract. We need to be extremely careful to subtract the ENTIRE expression, and not just its first term. To do that, we need to insert parentheses around these expressions.

$$\frac{(x+y) - (x-y)}{(x-y)(x+y)} = \frac{x+y-x+y}{(x-y)(x+y)} = \frac{2y}{(x-y)(x+y)} \text{ or } \frac{2y}{x^2 - y^2}$$

The final answer can be presented as $\boxed{\frac{2y}{(x-y)(x+y)} \text{ or } \frac{2y}{x^2 - y^2}}$. They are both equally correct.

$$4. \frac{2}{p-5} - \frac{p+11}{p^2-2p-15}$$

Solution: The denominator of the second fraction factors. $p^2 - 2p - 15 = (p+3)(p-5)$. We will now bring these fractions to the common denominator.

$$\frac{2}{p-5} - \frac{p+11}{p^2-2p-15} = \frac{2(p+3)}{(p-5)(p+3)} - \frac{p+11}{(p-5)(p+3)}$$

We can now perform the subtraction of polynomials in the numerator

$$\frac{2(p+3) - (p+11)}{(p-5)(p+3)} = \frac{2p+6-p-11}{(p-5)(p+3)} = \frac{p-5}{(p-5)(p+3)} = \frac{\cancel{p-5}}{\cancel{(p-5)}(p+3)} = \boxed{\frac{1}{p+3}}$$

$$5. \frac{x^2-5x+78}{18x+x^2-208} - \frac{x}{x+26}$$

Solution: The denominator of the first fraction factors. $x^2 + 18x - 208 = (x+26)(x-8)$. We will now bring these fractions to the common denominator.

$$\frac{x^2-5x+78}{x^2+18x-208} - \frac{x}{x+26} = \frac{x^2-5x+78}{(x+26)(x-8)} - \frac{x(x-8)}{(x+26)(x-8)}$$

We can now perform the subtraction of polynomials in the numerator

$$\frac{(x^2-5x+78) - x(x-8)}{(x+26)(x-8)} = \frac{x^2-5x+78-x^2+8x}{(x+26)(x-8)} = \frac{3x+78}{(x+26)(x-8)}$$

After we factor out the greatest common factor from the numerator, the result can be simplified.

$$\frac{3x+78}{(x+26)(x-8)} = \frac{3(\cancel{x+26})}{(\cancel{x+26})(x-8)} = \boxed{\frac{3}{x-8}}$$

$$6. \left[\left(\frac{1}{b} - b \right) \div \left(1 - \frac{1}{b} \right) \right] (1+b)$$

Solution: Let us perform the subtractions first.

$$\frac{1}{b} - b = \frac{1}{b} - \frac{b}{1} = \frac{1}{b} - \frac{b \cdot b}{1 \cdot b} = \frac{1}{b} - \frac{b^2}{b} = \frac{1-b^2}{b} \quad \text{and} \quad 1 - \frac{1}{b} = \frac{1}{1} - \frac{1}{b} = \frac{1 \cdot b}{1 \cdot b} - \frac{1}{b} = \frac{b}{b} - \frac{1}{b} = \frac{b-1}{b}$$

$$\text{Thus } \left[\left(\frac{1}{b} - b \right) \div \left(1 - \frac{1}{b} \right) \right] (1+b) = \left[\frac{1-b^2}{b} \div \frac{b-1}{b} \right] (1+b)$$

We re-write the division as multiplication by the reciprocal, and then cancel.

$$\left[\frac{1-b^2}{\cancel{b}} \cdot \frac{\cancel{b}}{b-1} \right] (1+b) = \frac{1-b^2}{b-1} (1+b)$$

This expression can be further simplified, since $1-b^2 = (1+b)(1-b)$ via the difference of squares theorem, and $1-b$ and $b-1$ are opposites.

$$\frac{1-b^2}{b-1} (1+b) = \frac{(1-b)(1+b)}{b-1} (1+b) = \frac{-1(b-1)(1+b)}{b-1} (1+b) = \frac{-1(\cancel{b-1})(1+b)}{\cancel{b-1}} (1+b) = \boxed{-(b+1)^2}$$

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