

Linear expression such as $2x + 1$ have no smallest or greatest values. Substituting greater and greater positive numbers (as x) into $2x + 1$ produces greater and greater values. Similarly, substituting large negative numbers (as x) into $2x + 1$ produces negative values with increasing absolute value.

Quadratic expressions are fundamentally different, as no square of any real number is negative. Consequently, most quadratic expressions have a smallest possible value: zero.

Example 1. What is the smallest value of each of the following quadratic expressions?

- a) x^2 b) $(x - 4)^2$ c) $(2x + 1)^2$ d) $(x + 7)^2$

- Solution:** a) No real number can have a negative square. Consequently, if x^2 could take the value zero, that would be the smallest value. That is possible, when we square $x = 0$. Thus, the lowest value of x^2 is zero, when x is zero.
- b) Recall that $(x - 4)^2$ is a complete square; a difference squared. No matter what the value of x is, $x - 4$ is a real number and so it can NOT have a negative square. Consequently, if $(x - 4)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(x - 4)^2$ is zero, when $x - 4$ is zero, that is, when $x = 4$. (Just solve the linear equation $x - 4 = 0$). In short: the lowest value of $(x - 4)^2$ is zero, when $x = 4$.
- c) Recall that $(2x + 1)^2$ is a square, thus no value for x will ever result in a negative value. Consequently, if $(2x + 1)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(2x + 1)^2$ is zero, when $2x + 1 = 0$. We solve the linear equation $2x + 1 = 0$ for x and obtain $x = -\frac{1}{2}$. So, the lowest value of $(2x + 1)^2$ is zero, when $x = -\frac{1}{2}$.
- d) Since $(x + 7)^2$ is a square, no value for x will ever result in a negative value. Consequently, if $(x + 7)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(x + 7)^2$ is zero, when $x + 7 = 0$. We solve the linear equation $x + 7 = 0$ for x and obtain $x = -7$. So, the lowest value of $(x + 7)^2$ is zero, when $x = -7$.

Example 2. Find the smallest value of each of the following quadratic expressions.

- a) $x^2 + 25$ b) $(x - 4)^2 - 1$ c) $(2x + 1)^2 + 12$ d) $(x + 7)^2 - 100$

- Solution:** First imagine a room where some people gathered. Everyone empties their vallets and pockets and count all their cash and then compare. As it turns out, Mr. X has the least amount of money on them, only \$1.50. Then another hour later, everyone in the room receives \$20. Who has the least amount of money now? The answer is clearly Mr. X, this time with \$21.50.
- a) Consider the expression $x^2 + 25$. Recall that the lowest value of x^2 is zero, when x is zero. Then the lowest value of $x^2 + 25$ is 25, when x is zero. (Imagine that the 'poorest' person in the room was Ms. Y, with no money on her. If then everyone in the room receives \$25, she would still be the one with the smallest amount of money, exactly \$20.)
- b) Consider now the expression $(x - 4)^2 - 1$. The lowest value of $(x - 4)^2$ is zero, when $x = 4$. Then the lowest value of $(x - 4)^2 - 1$ is -1 , when $x = 4$.
- c) Consider now the expression $(2x + 1)^2 + 12$. Because it is a square, the lowest value of $(2x + 1)^2$ is zero, when $x = -\frac{1}{2}$. Then the lowest value of $(2x + 1)^2 + 12$ is 12, when $x = -\frac{1}{2}$.
- d) Consider now the expression $(x + 7)^2 - 100$. Because it is a square, the lowest value of $(x + 7)^2$ is zero, when $x = -7$. Then the lowest value of $(x + 7)^2 - 100$ is -100 , when $x = -7$.

Example 3. Find the smallest value of each of the following quadratic expressions.

a) $x^2 + 2x - 5$ b) $x^2 - 8x + 15$ c) $x^2 - 12x$ d) $x^2 - 100$

Solution: a) We will apply the same ideas, but first we need to transform this expression into a more suitable form. For that, We simply complete the square. Half of the linear coefficient is 1, so our suitable complete square is $(x + 1)^2$.

$$\begin{aligned} E &= x^2 + 2x - 5 && \text{helper line: } (x + 1)^2 = x^2 + 2x + 1 \\ &= \underbrace{x^2 + 2x + 1}_{(x+1)^2} - 1 - 5 && \text{so we smuggle in 1} \\ &= (x + 1)^2 - 6 \end{aligned}$$

This form of a quadratic expression is often called **the standard form**. Once we brought the expression to the standard form, we can easily determine the smallest value. The smallest value of $(x + 1)^2 - 6$ is -6 , when $x = -1$.

b) Consider the expression $x^2 - 8x + 15$. We complete the square to bring the expression to its standard form.

$$\begin{aligned} E &= x^2 - 8x + 15 && \text{helper line: } (x - 4)^2 = x^2 - 8x + 16 \\ &= \underbrace{x^2 - 8x + 16}_{(x-4)^2} - 16 + 15 && \text{so we smuggle in 16} \\ &= (x - 4)^2 - 1 \end{aligned}$$

Once we brought the expression to the standard form, we can easily determine the smallest value. The smallest value of $(x - 4)^2 - 1$ is -1 , when $x = 4$.

c) Consider the expression $x^2 - 12x$. We complete the square to bring the expression to its standard form.

$$\begin{aligned} E &= x^2 - 12x && \text{helper line: } (x - 6)^2 = x^2 - 12x + 36 \\ &= \underbrace{x^2 - 12x + 36}_{(x-6)^2} - 36 && \text{so we smuggle in 36} \\ &= (x - 6)^2 - 36 \end{aligned}$$

The smallest value of $x^2 - 12x$ is -36 , when $x = 6$.

d) Consider the expression $x^2 - 100$. We do not complete the square as the expression is already in its standard form. If it helps, we can think of $x^2 - 100$ as $(x - 0)^2 - 100$. Either way, the smallest value of $x^2 - 100$ is -100 , when $x = 0$.

Completing the square is not just a factoring technique. It is really our only way (at this point) to determine the smallest value that the expression takes. Completing the square is a way to understand quadratic expressions.

Example 4v. Find the smallest value of each of the following quadratic expressions.

a) $x^2 - 10x + 14$ b) $(2a - 1)^2 + 8$ c) $x^2 - 8x - 180$ [Solution - Youtube link](#)

Example 5v. One number is 10 greater than another number. Find the smallest possible value of the product of these numbers. [Solution - Youtube link](#)

Solution: If we label the smaller number by x , then the other number is $x + 10$. The product of these numbers is $x(x + 10) = x^2 + 10x$, a quadratic expression. We complete the square to find the smallest possible value.

$$\begin{aligned} x(x + 10) &= x^2 + 10x & (x + 5)^2 &= x^2 + 10x + 25 \\ &= x^2 + 10x + 25 - 25 \\ &= (x + 5)^2 - 25 \end{aligned}$$

The smallest possible value of the complete square, $(x + 5)^2$ is zero. Then the smallest value of the entire expression is $\boxed{-25}$.

We can check: to achieve this smallest value, x must be -5 . Then the greater number is $-5 + 10 = 5$. Indeed, the product of 5 and -5 is -25 . Can we check if it is the smallest value? If we select another number for x , say -2 , then the greater number must be 8, and therefore the product is $-2 \cdot 8 = -16$. If x is positive, the other number is also positive, therefore, so is the product of the two numbers. So it does look like -25 is indeed the smallest value that this expression can achieve.

Part 2 - Optional Topic - The Arithmetic and Geometric Means

The fact that squares are non-negative and therefore have a smallest value pops up many times in mathematics. The following is an example for that. What we call average, is just one of several types of averages used in mathematics. The simplest one is the one we know simply as average, and it is called the arithmetic mean. There is also the geometric mean. As the name suggests, this mean shows up a lot in geometry. The harmonic means is an average that often shows up in physics.

Definition: Let a and b represent non-negative numbers. The arithmetic and geometric, means of a and b are defined as follows.

$$\text{arithmetic mean} = \frac{a + b}{2} \quad \text{and} \quad \text{geometric mean} = \sqrt{ab}$$

The harmonic average of a and b is defined as follows. Take the reciprocal of a and b . Take the 'regular' average of these reciprocals and then take the reciprocal of that. So the harmonic average of a and b is the reciprocal of $\frac{\frac{1}{a} + \frac{1}{b}}{2}$, which is $\frac{2}{\frac{1}{a} + \frac{1}{b}}$. Today we will only focus on the arithmetic and geometric means.

Example 6. Find the arithmetic, geometric, and harmonic means of 2 and 18.

Solution: The arithmetic mean is $\frac{2 + 18}{2} = 10$. The geometric mean is $\sqrt{2 \cdot 18} = 6$. The harmonic mean is (the reciprocal of the average of the reciprocals) is $\frac{2}{\frac{1}{2} + \frac{1}{18}} = \frac{2}{\frac{9}{18} + \frac{1}{18}} = \frac{2}{\frac{10}{18}} = 2 \cdot \frac{18}{10} = 3.6$

For any a and b , these means have a natural order. The arithmetic mean is always greater than or equal to the geometric mean: $\frac{a+b}{2} \geq \sqrt{ab}$ and the equality holds if and only if $a = b$. Why would this be so? Because squares are non-negative, and only zero when we square zero.

Theorem: (The arithmetic and geometric means). Suppose that a and b are positive numbers. Then

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and the equality holds if and only if $a = b$.

Proof: Recall that a and b are both positive, so $\sqrt{a}$ and $\sqrt{b}$ both exist. For all a and b , $(\sqrt{a} - \sqrt{b})^2 \geq 0$ and the equality holds if and only if $a = b$.

$$\begin{aligned} (\sqrt{a} - \sqrt{b})^2 &\geq 0 \\ (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0 && \text{add } 2\sqrt{ab} \\ a + b &\geq 2\sqrt{ab} && \text{divide by 2} \\ \frac{a+b}{2} &\geq \sqrt{ab} \end{aligned}$$

We prove inequalities too, not just solve them. In geometry, many inequalities we prove boil down to either the triangle inequality, or the arithmetic-geometric means theorem. When we look at an inequality we want to prove, the type of the inequality itself is an important clue. Strict inequalities (i.e $>$ or greater) suggest the triangle inequality. When we see $\geq$ greater than or equal in the relationship to be proved, chances are that we can prove it using the arithmetic-geometric means.

Example 7. Prove that for all positive numbers x , $x + \frac{1}{x} \geq 2$.

Solution: Let us apply the arithmetic-geometric means to the positive numbers x and $\frac{1}{x}$. The arithmetic mean of these

two numbers is $\frac{x + \frac{1}{x}}{2}$, and their geometric mean is $\sqrt{x \cdot \frac{1}{x}} = \sqrt{1} = 1$. Now we state the inequality:

$$\begin{aligned} \frac{x + \frac{1}{x}}{2} &\geq 1 && \text{multiply by 2} \\ x + \frac{1}{x} &\geq 2 \end{aligned}$$

This completes our proof.



Practice Problems

Find the smallest value of each of the following expressions.

1. $x^2 - 4x + 7$
2. $x^2 + 10x + 14$
3. $x^2 - 2x$
4. $x^2 + 6x + 42$
5. $x^2 - 100x + 60$
6. $x^2 + 8x + 20$
7. $x^2 - 16$
8. $x^2 - 16x + 4$
9. $x^2 - 4x + 4$
10. $x^2 + 18x + 81$
11. One number is 24 less than another. Find the smallest possible value of the product of these numbers.



Enrichment

1. Not all squares produce zero as their smallest value. Consider the expression $(x^2 + 3)^2$. What is the smallest value of this expression? How about $(x^2 + 3)^2 - 1$? And $x^4 + 10x^2 + 14$?
2. Until now, the quadratic expressions we saw all had a leading coefficient of 1. How is our argument modified with different leading coefficients? Discuss the smallest value of each of the given expressions.
 - a) $3(x - 5)^2 + 8$
 - b) $-2(x + 1)^2 - 49$
 - c) $5((x + 4)^2 - 25)$
3. The product of two positive numbers is 6. Find the smallest possible value of the sum of these two numbers.



Answers

Practice Problems

1. 3 2. -11 3. -1 4. 33 5. -2440 6. 4 7. -16 8. -60 9. 0 10. 0 11. -144