

Definition: A function  $f$  is **one-to-one (or injective)** if for all  $a$  and  $b$  in its domain, if  $a \neq b$ , then  $f(a) \neq f(b)$ .

Alternative definition: A function  $f$  is **one-to-one (or injective)** if for all  $a$  and  $b$  in its domain, if  $f(a) = f(b)$ , then  $a = b$ .

Definition: A function  $f$  is **increasing** on an interval  $I$  if for all  $a$  and  $b$  in  $I$ , if  $a < b$ , then  $f(a) \leq f(b)$ .

Definition: A function  $f$  is **strictly increasing** on an interval  $I$  if for all  $a$  and  $b$  in  $I$ , if  $a < b$ , then  $f(a) < f(b)$ .

Definition: A function  $f$  is **decreasing** on an interval  $I$  if for all  $a$  and  $b$  in  $I$ , if  $a < b$ , then  $f(a) \geq f(b)$ .

Definition: A function  $f$  is **strictly decreasing** on an interval  $I$  if for all  $a$  and  $b$  in  $I$ , if  $a < b$ , then  $f(a) > f(b)$ .

Definition: A function  $f$  is **even** if for all  $x$  in its domain,  $f(-x) = f(x)$ . The graph of an even function is symmetrical to the  $y$ -axis.

Definition: A function  $f$  is **odd** if for all  $x$  in its domain,  $f(-x) = -f(x)$ . The graph of an odd function is symmetrical to the origin.

Note that while an integer is either even or odd, most functions are neither even, nor odd. Even and odd functions are sort of rare but these properties are very useful for us.

Definition: A **rational function** is a quotient of two polynomial functions.

The concept of a **continuous function** is very important. Although this term will not be precisely defined, the intuitive idea of a continuous function is that we can draw its graph without lifting the pencil.

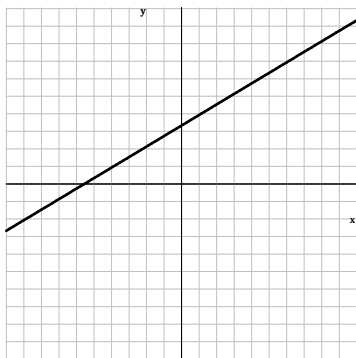
For example,  $f(x) = x^2$  is a continuous function but  $g(x) = \frac{1}{x}$  is not; it is not continuous at  $x = 0$ .

## Basic Functions

1.) Linear functions  $f(x) = mx + b$  where  $m \neq 0$ .

The graph is a straight line. One useful form of the equation can be obtained by factoring out the slope:

$$f(x) = mx + b = m \left( x + \frac{b}{m} \right)$$



$$m > 0$$

Case 1. If  $m > 0$

domain:  $\mathbb{R}$

range:  $\mathbb{R}$

$y$ -intercept:  $(0, b)$

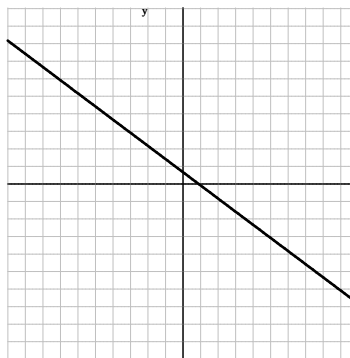
$x$ -intercept:  $\left(-\frac{b}{m}, 0\right)$

one-to-one

no maximum or minimum

strictly increasing

continuous on  $\mathbb{R}$



$$m < 0$$

Case 2. If  $m < 0$

domain:  $\mathbb{R}$

range:  $\mathbb{R}$

$y$ -intercept:  $(0, b)$

$x$ -intercept:  $\left(-\frac{b}{m}, 0\right)$

one-to-one

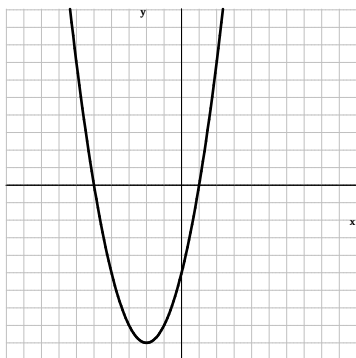
no maximum or minimum

strictly decreasing

continuous on  $\mathbb{R}$

2.) Quadratic functions  $f(x) = ax^2 + bx + c$  where  $a \neq 0$ .

The graph is a parabola. It opens upward if  $a > 0$  and opens downward if  $a < 0$ .



Case 1.  $a > 0$   
If  $a > 0$

Example:  $f(x) = x^2 + 4x - 5$

standard form:  $f(x) = (x + 2)^2 - 9$

factored form:  $f(x) = (x + 5)(x - 1)$

domain:  $\mathbb{R}$  range:  $[-9, \infty)$

$y$ -intercept:  $(0, -5)$

$x$ -intercepts:  $(-5, 0)$  and  $(1, 0)$

not one-to-one

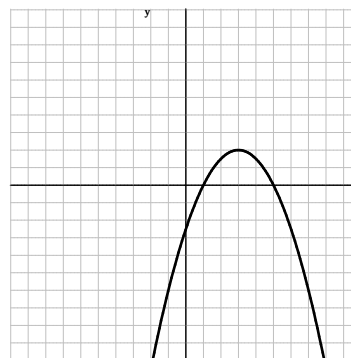
no maximum

minimum:  $(-2, -9)$

strictly decreasing on  $(-\infty, -2)$

strictly increasing on  $(-2, \infty)$

continuous on  $\mathbb{R}$



Case 2.  $a < 0$   
If  $a < 0$

Example:  $f(x) = -\frac{1}{2}x^2 + 3x - \frac{5}{2}$

standard form:  $f(x) = -\frac{1}{2}(x - 3)^2 + 2$

factored form:  $f(x) = -\frac{1}{2}(x - 1)(x - 5)$

domain:  $\mathbb{R}$  range:  $(-\infty, 2]$

$y$ -intercept:  $(0, -\frac{5}{2})$

$x$ -intercepts:  $(1, 0)$  and  $(5, 0)$

not one-to-one

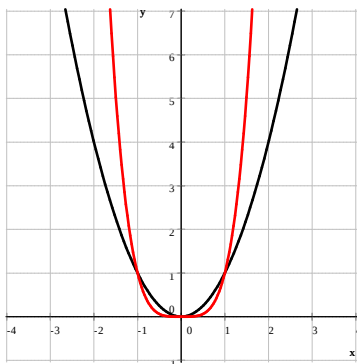
no minimum

maximum:  $(3, 2)$

strictly increasing on  $(-\infty, 3)$

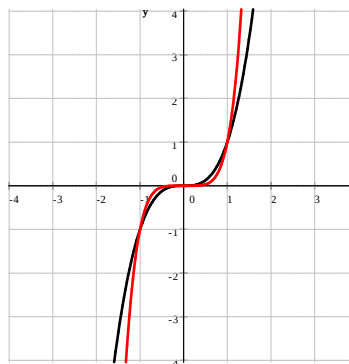
strictly decreasing on  $(3, \infty)$

continuous on  $\mathbb{R}$

3.) Monomials  $f(x) = x^n$  $n$  is evenCase 1. If  $n$  is evendomain:  $\mathbb{R}$  range:  $[0, \infty)$  $y$ -intercept:  $(0, 0)$  $x$ -intercept:  $(0, 0)$ 

not one-to-one

no maximum

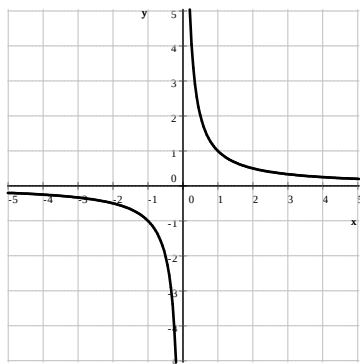
minimum:  $(0, 0)$ strictly decreasing on  $(-\infty, 0)$ strictly increasing on  $(0, \infty)$ continuous on  $\mathbb{R}$ black graph:  $f(x) = x^2$ red graph:  $f(x) = x^4$  $n$  is oddCase 2. If  $n$  is odddomain:  $\mathbb{R}$  range:  $\mathbb{R}$  $y$ -intercept:  $(0, 0)$  $x$ -intercept:  $(0, 0)$ 

one-to-one

no minimum or maximum

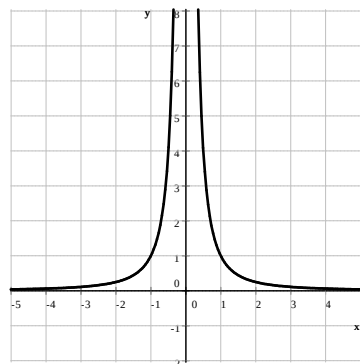
strictly increasing on  $\mathbb{R}$ continuous on  $\mathbb{R}$ black graph:  $f(x) = x^3$ red graph:  $f(x) = x^5$

4.) The rational functions  $f(x) = \frac{1}{x}$  and  $g(x) = \frac{1}{x^2}$



$$f(x) = \frac{1}{x}$$

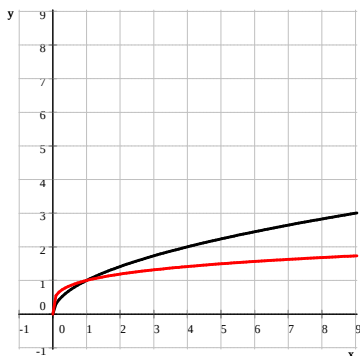
domain:  $\mathbb{R} \setminus \{0\}$  range:  $\mathbb{R} \setminus \{0\}$   
no  $y$ -intercept  
no  $x$ -intercept  
one-to-one  
no maximum or minimum  
strictly decreasing on  $(-\infty, 0)$  and on  $(0, \infty)$   
  
not continuous at  $x = 0$   
vertical asymptote: the line  $x = 0$   
horizontal asymptote: the line  $y = 0$



$$g(x) = \frac{1}{x^2}$$

domain:  $\mathbb{R} \setminus \{0\}$  range:  $(0, \infty)$   
no  $y$ -intercept  
no  $x$ -intercept  
not one-to-one  
no minimum or maximum  
strictly increasing on  $(-\infty, 0)$  and  
strictly decreasing on  $(0, \infty)$   
not continuous at  $x = 0$   
vertical asymptote: the line  $x = 0$   
horizontal asymptote: the line  $y = 0$

5.) Radical functions  $f(x) = \sqrt[n]{x}$



$n$  is even

Case 1. If  $n$  is even

domain:  $[0, \infty)$  range:  $[0, \infty)$

$y$ -intercept:  $(0, 0)$

$x$ -intercept:  $(0, 0)$

one-to-one

no maximum

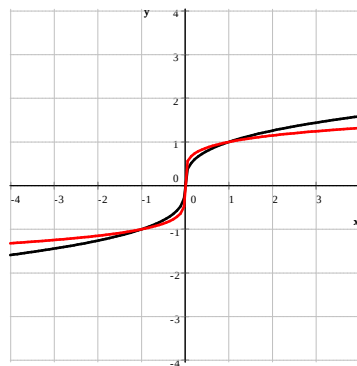
minimum:  $(0, 0)$

strictly increasing

continuous on  $(0, \infty)$

black graph:  $f(x) = \sqrt{x}$

red graph:  $f(x) = \sqrt[4]{x}$



$n$  is odd

Case 2. If  $n$  is odd

domain:  $\mathbb{R}$  range:  $\mathbb{R}$

$y$ -intercept:  $(0, 0)$

$x$ -intercept:  $(0, 0)$

one-to-one

no minimum or maximum

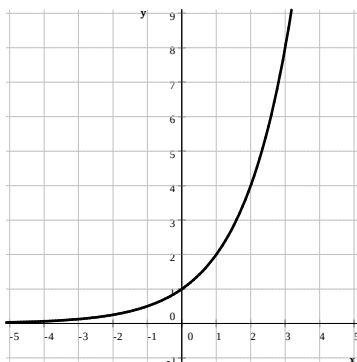
strictly increasing

continuous on  $\mathbb{R}$

black graph:  $f(x) = \sqrt[3]{x}$

red graph:  $f(x) = \sqrt[5]{x}$

6.) Exponential functions  $f(x) = a^x$  where  $a > 0$ .



$$a > 1$$

Case 1. If  $a > 1$

domain:  $\mathbb{R}$  range:  $(0, \infty)$

no  $x$ -intercepts

$y$ -intercept:  $(0, 1)$

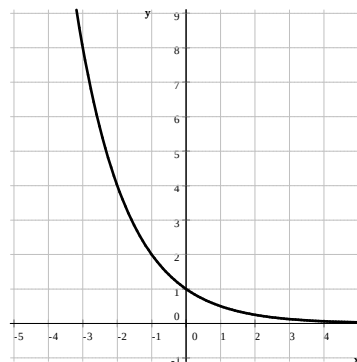
one-to-one

no maximum or minimum

strictly increasing

horizontal asymptote:  $y = 0$

continuous on  $\mathbb{R}$



Case 2. If  $0 < a < 1$

domain:  $\mathbb{R}$  range:  $(0, \infty)$

no  $x$ -intercepts

$y$ -intercept:  $(0, 1)$

one-to-one

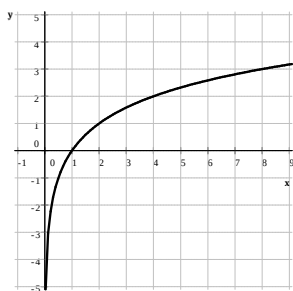
no maximum or minimum

strictly decreasing

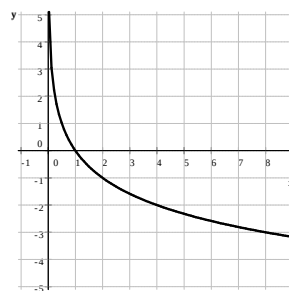
horizontal asymptote:  $y = 0$

continuous on  $\mathbb{R}$

7.) Logarithmic functions  $f(x) = \log_a x$  where  $a > 0$  and  $a \neq 1$ .

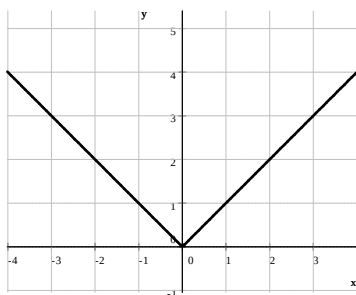


$a > 1$   
 Case 1. If  $a > 1$   
 domain:  $(0, \infty)$  range:  $\mathbb{R}$   
 no  $y$ -intercepts  
 $x$ -intercept:  $(1, 0)$   
 one-to-one  
 no maximum or minimum  
 strictly increasing  
 vertical asymptote:  $x = 0$   
 continuous on  $(0, \infty)$



$0 < a < 1$   
 Case 2. If  $0 < a < 1$   
 domain:  $(0, \infty)$  range:  $\mathbb{R}$   
 no  $y$ -intercepts  
 $x$ -intercept:  $(1, 0)$   
 one-to-one  
 no maximum or minimum  
 strictly decreasing  
 vertical asymptote:  $x = 0$   
 continuous on  $(0, \infty)$

8.) Absolute value function,  $f(x) = |x|$



domain:  $\mathbb{R}$  range:  $[0, \infty)$   
 $x$ -intercept:  $(0, 0)$ ,  $y$ -intercept:  $(0, 0)$   
 not one-to-one  
 no maximum  
 minimum:  $(0, 0)$   
 strictly decreasing on  $(-\infty, 0)$  and strictly increasing on  $(0, \infty)$   
 continuous on  $\mathbb{R}$

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