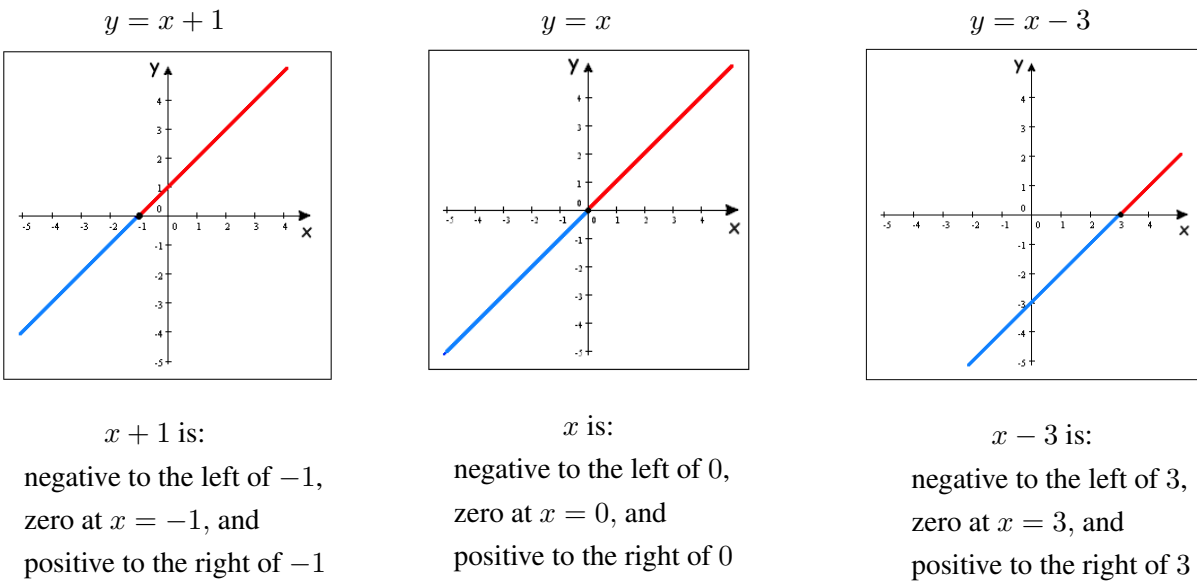


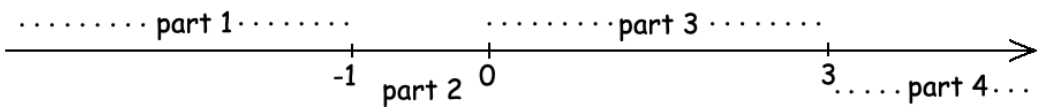
In case of higher degree inequalities, some of them can be easily handled. We will be looking at inequalities in which one side is zero, and the other side is factored. In such cases, all we do is count the negative factors. An even number of negative factors mean a positive product, an odd number of negative factors mean a negative product.

Example 1. Solve the inequality $(x + 1) x (x - 3) \geq 0$.

Solution: Let us first note that the behavior of each linear factor is quite simple. A line with a positive slope is (left to right) negative first, then takes a zero, and then is positive forever. This can be summarized by a graph:



This is all the information we need to determine the sign of the expression $(x + 1) x (x - 3)$. We will organize this data in the table shown below. The number line will be divided into four parts by the three points $x = -1$, $x = 0$, and $x = 3$ as the picture below indicates.



We indicate these four intervals horizontally, and the three factors in each row as shown below. The fourth row will be for the product $(x + 1) x (x - 3)$.

	on $(-\infty, -1)$	on $(-1, 0)$	on $(0, 3)$	on $(3, \infty)$
the factor $(x + 1)$				
the factor x				
the factor $x - 3$				
the product $(x + 1) x (x - 3)$				

We fill out the first row, indicating the sign of $x + 1$ on each interval. Recall that this linear expression is negative before -1 and positive after.

	on $(-\infty, -1)$	on $(-1, 0)$	on $(0, 3)$	on $(3, \infty)$
the factor $(x + 1)$	—	+	+	+
the factor x				
the factor $x - 3$				
the product $(x + 1)x(x - 3)$				

We fill out the second row, indicating the sign of x on each interval. Recall that this linear expression is negative before 0 and positive after.

	on $(-\infty, -1)$	on $(-1, 0)$	on $(0, 3)$	on $(3, \infty)$
the factor $(x + 1)$	—	+	+	+
the factor x	—	—	+	+
the factor $x - 3$				
the product $(x + 1)x(x - 3)$				

Next, we fill out the third row, indicating the sign of $x - 3$ on each interval. Recall that this linear expression is negative before 3 and positive after.

	on $(-\infty, -1)$	on $(-1, 0)$	on $(0, 3)$	on $(3, \infty)$
the factor $(x + 1)$	—	+	+	+
the factor x	—	—	+	+
the factor $x - 3$	—	—	—	+
the product $(x + 1)x(x - 3)$				

	on $(-\infty, -1)$
the factor $(x + 1)$	—
the factor x	—
the factor $x - 3$	—
the product $(x + 1)x(x - 3)$	—

We are now ready to determine the sign of the product $(x + 1)x(x - 3)$ on each of the intervals. On the interval $(-\infty, -1)$ the product is negative because we are multiplying three negative numbers, as each factor is negative.

On the interval $(-1, 0)$ the product is positive because we are multiplying two negative and a positive number, as indicated by the second column. On the interval $(0, 3)$, we are multiplying two positive and a negative number, so the product is negative. Finally, the product of three positive numbers is positive, and so the expression $(x + 1)x(x - 3)$ is positive on $(3, \infty)$. We fill out the last row accordingly and obtain the table shown below.

	on $(-\infty, -1)$	on $(-1, 0)$	on $(0, 3)$	on $(3, \infty)$
the factor $(x + 1)$	—	+	+	+
the factor x	—	—	+	+
the factor $x - 3$	—	—	—	+
the product $(x + 1)x(x - 3)$	—	+	—	+

We now read the information from the last row: the product $(x + 1)x(x - 3)$ is negative on $(-\infty, -1)$ and on $(0, 3)$ and is positive on $(-1, 0)$ and $(3, \infty)$.

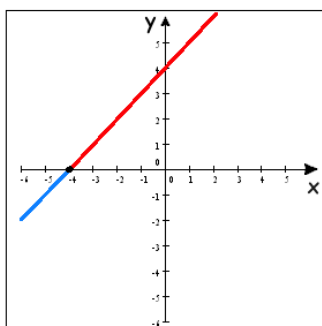
So the inequality $(x + 1)x(x - 3) \geq 0$ has solution set, in interval notation: $[-1, 0] \cup [3, \infty)$.

Example 2. Solve the inequality $-3(x + 4)(x - 2)^2 < 0$.

Solution: We will determine the sign of each factor and then determine the sign of the product by counting the negative factors.

Consider the graphs $y = x + 4$ and $y = (x - 2)^2$. The line $y = x + 4$ is negative before $x = -4$, is zero at $x = -4$, and then it is positive after -4 . The parabola $y = (x - 2)^2$ is almost always positive, except for $x = 2$, when it is zero. We will consider the coefficient -3 as a separate factor that is always negative.

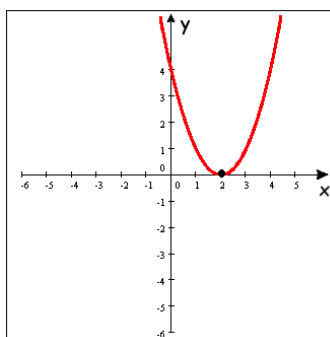
$$y = x + 4$$



$$x + 4 \text{ is:}$$

negative to the left of -4 ,
zero at $x = -4$, and
positive to the right of -4

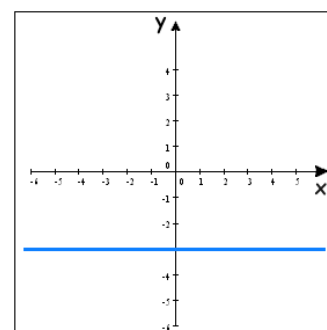
$$y = (x - 2)^2$$



$$(x - 2)^2 \text{ is:}$$

positive to the left of 2,
zero at $x = 2$, and
positive to the right of 2

$$y = -3$$



$$-3 \text{ is:}$$

negative everywhere.

This is all the information we need to determine the sign of the expression $-3(x + 4)(x - 2)^2$. We will organize this data in the table shown below. The number line will be divided into three parts by the two points $x = -4$ and $x = 2$.

We indicate these three intervals $(-\infty, -4)$, $(-4, 2)$, and $(2, \infty)$ in the top row. We indicate the three factors in each row as shown below. The fourth row will be for the product $-3(x + 4)(x - 2)^2$. We indicate the sign of each factor in each interval.

	on $(-\infty, -4)$	on $(-4, 2)$	on $(2, \infty)$
the factor $x + 4$	—	+	+
the factor $(x - 2)^2$	+	+	+
the factor -3	—	—	—
the product $-3(x + 4)(x - 2)^2$	+	—	—

We now read the information from the last row: the product $-3(x+4)(x-2)^2$ is positive on $(-\infty, -4)$ and negative on $(-4, 2)$ and $(2, \infty)$. Thus the solution of $-3(x+4)(x-2)^2 < 0$ is $(-4, 2) \cup (2, \infty)$.

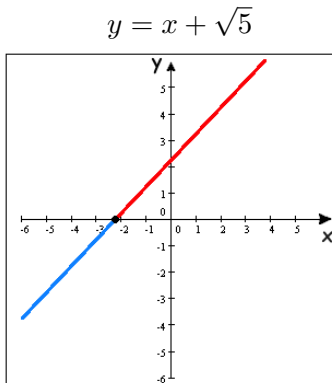
Example 3. Solve the inequality $5x - x^3 \leq 0$.

Solution: Until now, the expressions were given in a factored form. This time we will need to factor the polynomial given. We start by the leading coefficient, -1 .

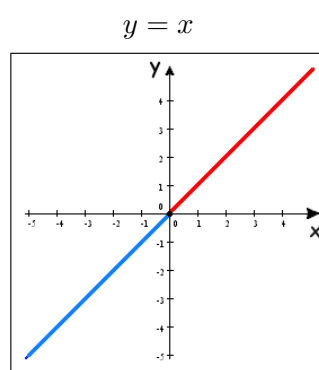
$$\begin{aligned} 5x - x^3 &= -(x^3 - 5x) && \text{factor out } x \\ &= -x(x^2 - 5) && \text{factor via the difference of squares theorem} \\ &= -x(x + \sqrt{5})(x - \sqrt{5}) \end{aligned}$$

By the zero product rule, the equation $-x(x + \sqrt{5})(x - \sqrt{5}) = 0$ has three solutions: $-\sqrt{5}$, 0 , and $\sqrt{5}$.

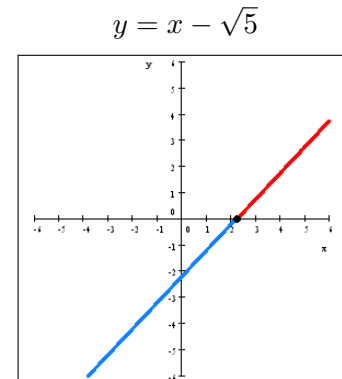
We will determine the sign of each factor and then determine the sign of the product by counting the negative signs. Consider the graphs of $y = x + \sqrt{5}$, $y = x$, and $y = x - \sqrt{5}$. These are all lines, negative before the zero at $-\sqrt{5}$, 0 , and $\sqrt{5}$, respectively, and positive after.



$x + \sqrt{5}$ is:
negative to the left of $-\sqrt{5}$,
zero at $x = -\sqrt{5}$, and
positive to the right of $-\sqrt{5}$



x is:
negative to the left of 0 ,
zero at $x = 0$, and
positive to the right of 0



$x - \sqrt{5}$ is:
negative to the left of $\sqrt{5}$,
zero at $x = \sqrt{5}$, and
positive to the right of $\sqrt{5}$

This is all the information we need to determine the sign of the expression $-x(x + \sqrt{5})(x - \sqrt{5})$. We will organize this data in the table shown below. We will look at this product as one of four factors: x , $x + \sqrt{5}$, $x - \sqrt{5}$, and the number -1 as a separate factor that is always negative. The number line will be divided into four parts by the three points $x = -\sqrt{5}$, $x = 0$, and $x = \sqrt{5}$. We indicate the four factors in each row as shown below. The fourth row will be for the product $-x(x + \sqrt{5})(x - \sqrt{5})$. We indicate the sign of each factor in each interval.

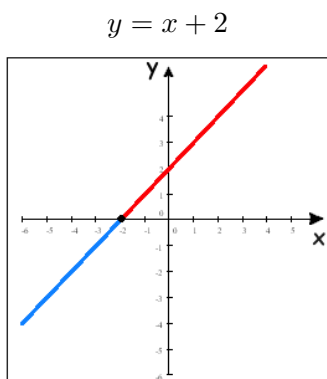
	on $(-\infty, -\sqrt{5})$	on $(-\sqrt{5}, 0)$	on $(0, \sqrt{5})$	on $(\sqrt{5}, \infty)$
the factor $x + \sqrt{5}$	—	+	+	+
the factor x	—	—	+	+
the factor $x - \sqrt{5}$	—	—	—	+
the factor -1	—	—	—	—
the product $-x(x + \sqrt{5})(x - \sqrt{5})$	+	—	+	—

We now read the information from the last row: the product is positive on $(-\infty, -\sqrt{5})$ and $(0, \sqrt{5})$, and is negative on $(-\sqrt{5}, 0)$ and $(\sqrt{5}, \infty)$. Thus the solution of $5x - x^3 \leq 0$ is $[-\sqrt{5}, 0] \cup [\sqrt{5}, \infty)$.

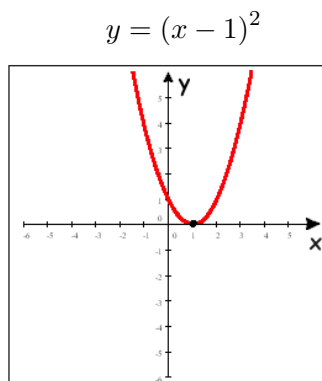
Example 4. Solve the inequality $(x + 2)(x - 1)^2(x - 3)^3 \geq 0$.

Solution: The equation $(x + 2)(x - 1)^2(x - 3)^3 = 0$ has three solutions: $x = -2$, $x = 1$, and $x = 3$.

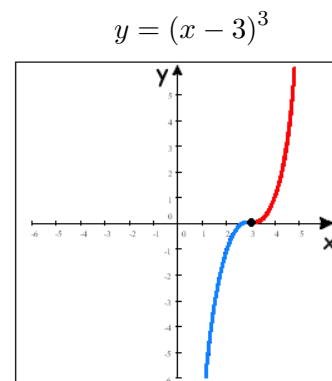
Consider the graphs of $y = x + 2$, $y = (x - 1)^2$, and $y = (x - 3)^3$.



$x + 2$ is:
negative to the left of -2 ,
zero at $x = -2$, and
positive to the right of -2



$(x - 1)^2$ is:
positive to the left of 1,
zero at $x = 1$, and
positive to the right of 1



$(x - 3)^3$ is:
negative to the left of 3,
zero at $x = 3$, and
positive to the right of 3

This is all the information we need to determine the sign of the expression $(x + 2)(x - 1)^2(x - 3)^3$. We will organize this data in the table shown below. We will look at this product as one of three factors: $(x + 2)$, $(x - 1)^2$, and $(x - 3)^3$. The number line will be divided into four parts by the three points $x = -2$, $x = 1$, and $x = 3$. We indicate the three factors in each row as shown below. The fourth row will be for the product $(x + 2)(x - 1)^2(x - 3)^3$. We indicate the sign of each factor in each interval.

	on $(-\infty, -2)$	on $(-2, 1)$	on $(1, 3)$	on $(3, \infty)$
the factor $x + 2$	—	+	+	+
the factor $(x - 1)^2$	+	+	+	+
the factor $(x - 3)^3$	—	—	—	+
the product $(x + 2)(x - 1)^2(x - 3)^3$	+	—	—	+

Based on the table, the product $(x + 2)(x - 1)^2(x - 3)^3$ is positive on $(-\infty, -2)$ and $(3, \infty)$, and zero when $x = -2$, 1, or 3. We need to be careful to include 1 in the solution set, which is therefore $(-\infty, -2] \cup \{1\} \cup [3, \infty)$.



Practice Problems

Solve each of the given inequalities.

1. $(x + 3)x(x - 2) < 0$

4. $(x - x^2 + 2)(x - 4) \geq 0$

7. $x^5 - x^3 > 0$

2. $9x - x^3 \leq 0$

5. $-2(x + 1)(x - 3)^2 < 0$,

8. $-x^3(x^2 - 4)^4 > 0$

3. $x^3 - 2x^2 - 8x < 0$

6. $(x + 2)^2x(x - 3)^3 \leq 0$

9. $-(x + 2)^2x(x - 3)^3 \leq 0$

10. $(x^2 - 6)(x - 1)^2(x + 1) \geq 0$

11. $x^5 - 10x^3 \leq 0$

12. $(x^2 - 9)(x^2 - 1) \geq 0$



Answers

1. $(-\infty, -3) \cup (0, 2)$
2. $[-3, 0] \cup [3, \infty)$
3. $(-\infty, -2) \cup (0, 4)$
4. $(-\infty, -1] \cup [2, 4]$
5. $(-1, 3) \cup (3, \infty)$
6. $\{-2\} \cup [0, 3]$
7. $(-1, 0) \cup (1, \infty)$
8. $(-\infty, -2) \cup (-2, 0)$
9. $(-\infty, 0] \cup [3, \infty)$
10. $[-\sqrt{6}, -1] \cup \{1\} \cup [\sqrt{6}, \infty)$
11. $(-\infty, -\sqrt{10}] \cup [0, \sqrt{10}]$
12. $(-\infty, -3] \cup [-1, 1] \cup [3, \infty)$

Caution! When entering the solution sets in MyOpenMath, a one-element set such as $\{-2\}$ must be entered as an interval: as $[-2, 2]$