

Definition: Let a and b represent non-negative numbers. The arithmetic and geometric, and harmonic means of a and b are defined as follows.

$$\text{arithmetic mean} = \frac{a+b}{2} \quad \text{and} \quad \text{geometric mean} = \sqrt{ab}$$

For any a and b , these means have a natural order. The arithmetic mean is always greater than or equal to the geometric mean: $\frac{a+b}{2} \geq \sqrt{ab}$ and the equality holds if and only if $a = b$.

Theorem: (The arithmetic and geometric means). Suppose that a and b are positive numbers. Then

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and the equality holds if and only if $a = b$.

Proof: Recall that a and b are both positive, so $\sqrt{a}$ and $\sqrt{b}$ both exist. For all a and b , $(\sqrt{a} - \sqrt{b})^2 \geq 0$ and the equality holds if and only if $a = b$.

$$\begin{aligned} (\sqrt{a} - \sqrt{b})^2 &\geq 0 \\ (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0 && \text{add } 2\sqrt{ab} \\ a + b &\geq 2\sqrt{ab} && \text{divide by 2} \\ \frac{a+b}{2} &\geq \sqrt{ab} \end{aligned}$$

The inequality between arithmetic and geometric means has many applications. Here is one example.

Example 1. Prove that for all positive numbers x , $x + \frac{1}{x} \geq 2$.

Solution: Let us apply the arithmetic-geometric means to the positive numbers x and $\frac{1}{x}$. The arithmetic mean of these two

numbers is $\frac{x + \frac{1}{x}}{2}$, and their geometric mean is $\sqrt{x \cdot \frac{1}{x}} = \sqrt{1} = 1$. Now we state the inequality:

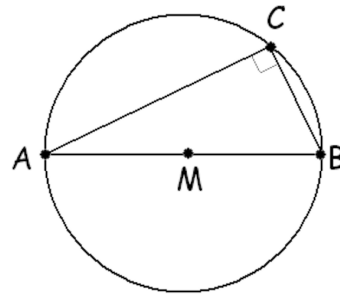
$$\begin{aligned} \frac{x + \frac{1}{x}}{2} &\geq 1 && \text{multiply by 2} \\ x + \frac{1}{x} &\geq 2 \end{aligned}$$

This completes our proof.

This theorem can be proved using just a picture. Suppose that a and b are two positive numbers. We will need two facts from geometry.

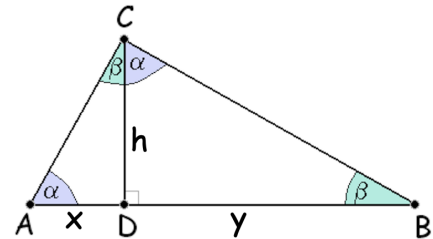
Claim 1. (Thales theorem)

Suppose that AB is a line segment and M is its midpoint. Suppose both A and B lie on a circle centered at M . (i.e. AB is a diameter of the circle) If C is any point on the circle different from A and B , then triangle ABC is a right triangle with the right angle at C . This circle is called the **Thales circle** of the line segment.



Claim 2. Let ABC be a right triangle with hypotenuse AB , and h the height drawn to the hypotenuse. Let x and y denote the two line segments into which the hypotenuse is split by h . Then h is the geometric mean of x and y , $h = \sqrt{xy}$.

Proof. Triangle ABC is a right triangle, and so $\alpha + \beta = 90^\circ$. Since triangle ADC has a right angle at D and α at A , the third angle, at C , must be β . Similarly, triangle DBC has a right angle at D and β at B , the third angle, at C , is forced to be α . Therefore, triangles ABC , CBD , and ACD are similar to each other. We will look at the ratio of the side opposite β to the side opposite α in the two smaller triangles.



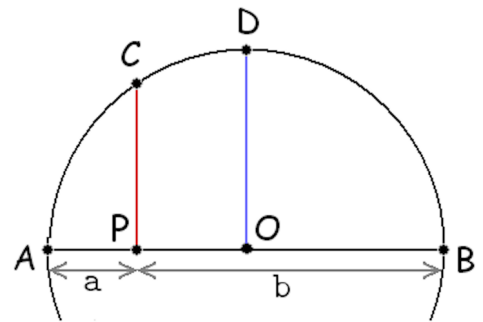
$$\frac{\text{side opposite } \beta}{\text{side opposite } \alpha} = \frac{h}{y} \text{ in triangle } BCD \quad \text{and} \quad \frac{\text{side opposite } \beta}{\text{side opposite } \alpha} = \frac{x}{h} \text{ in triangle } ACD$$

Therefore, we have that $\frac{h}{y} = \frac{x}{h} \implies h^2 = xy \implies h = \sqrt{xy}$ (we ruled out the negative solution of $h^2 = xy$ as distances cannot be negative)

We are ready to prove the arithmetic-geometric mean inequality using geometry.

Claim: If a and b are two positive numbers, then $\sqrt{ab} \leq \frac{a+b}{2}$ and the two sides are equal if and only if $a = b$.

Proof: Let us measure a and b on the same line, next to each other. The point separating them is P . Let us draw the Thales circle of line segment AB centered at O . We draw a line that is perpendicular to AB and passes through P . Let C be the intersection of this line with the circle. Since C is on the circle, triangle ABC is a right triangle, with the right angle at C . Line PC is the height drawn to the hypotenuse. Therefore, length of line segment PC is the geometric mean of the two pieces into which PC splits AB , in this case, a and b . In short, PC is the geometric mean of a and b , $PC = \sqrt{ab}$



Consider now line segment OD . It is clearly the radius of the circle, so its length is $\frac{a+b}{2}$. So, $OD = \frac{a+b}{2}$.

We can see on the picture that $PC \leq OD$, and the equality holds if and only if $a = b$.

So, $\sqrt{ab} \leq \frac{a+b}{2}$, and the equality holds if and only if $a = b$.

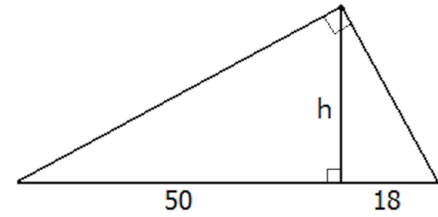
This completes our proof. We can also see that the greater the difference between a and b , the greater the difference is between

the arithmetic and geometric means.



Practice Problems

1. Find both means for $a = 36$ and $b = 64$.
2. The picture shows a right triangle. Find the length of the height drawn to the hypotenuse.
3. Suppose that x is a positive number. Find the smallest possible value of $x + \frac{9}{x}$.

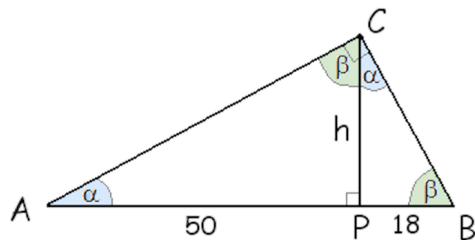
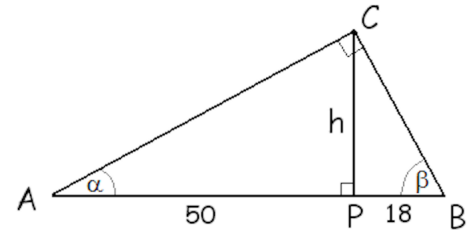




Answers

1. Arithmetic Mean: $\frac{36 + 64}{2} = 50$ and geometric Mean: $\sqrt{36 \cdot 64} = 48$

2. Solution: Let us first label the points, angles and sides in the triangle. Since ABC triangle is a right triangle, we have that $\alpha + \beta = 90^\circ$. Because of this, angle ACP must be equal to β , and angle PCB is equal to α . Thus the height drawn to the hypotenuse splits the original triangle into two triangles that have identical angles as the original triangle. Thus, all three triangles, $\triangle ABC$, $\triangle APC$ and $\triangle PBC$ are similar.



Consider now the ratio $\frac{\text{side opposite angle } \beta}{\text{side opposite angle } \alpha}$ in triangles $\triangle APC$ and $\triangle PBC$. Since these triangles are similar, this ratio is preserved.

$$\frac{\text{side opposite angle } \beta}{\text{side opposite angle } \alpha} = \frac{50}{h} = \frac{h}{18}$$

We solve this equation for h :

$$\begin{aligned} \frac{50}{h} &= \frac{h}{18} \\ 50 \cdot 18 &= h^2 \\ 900 &= h^2 \\ h &= \pm 30 \end{aligned}$$

$h = -30$ is ruled out since distances can not be negative. Thus $h = \sqrt{18 \cdot 50} = 30$, the geometric mean of 18 and 50.

3. Apply the arithmetic-geometric mean to x and $\frac{9}{x}$.

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\frac{x + \frac{9}{x}}{2} \geq \sqrt{x \cdot \frac{9}{x}}$$

$$\frac{x + \frac{9}{x}}{2} \geq 3$$

$$x + \frac{9}{x} \geq 6$$

$$\sqrt{x \cdot \frac{9}{x}} = \sqrt{9} = 3$$

multiply by 2

So the smallest possible value is 6, when $x = \frac{9}{x}$ which is when $x = 3$. ($x = -3$ is ruled out.)