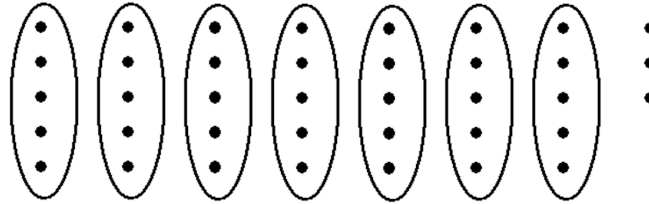


Theorem: (Division with Remainder) For every integers N and m , $m \neq 0$, there exist unique integers q and r such that

$$N = mq + r \quad \text{and} \quad r < m$$

For example, if $N = 38$ and $m = 5$, then the quotient is $q = 7$ and the remainder is $r = 3$. The picture below illustrates the division $38 \div 5 = 7 \text{ R } 3$, or, in other form: $38 = 5 \cdot 7 + 3$.



Note that there are two different ways to represent this division: $38 \div 5 = 7 \text{ R } 3$ and $\frac{38}{5} = 7\frac{3}{5}$

This theorem, also called the Euclidean division, is very fundamental to mathematics. There is a similar theorem about division of polynomials.

Theorem: (Division with Remainder for Polynomials) For every polynomials $N(x)$ and $m(x)$, $m(x)$ not the constant zero polynomial, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$N(x) = m(x)q(x) + r(x) \quad \text{and the degree of } r(x) \text{ is less than that of } m(x)$$

The process of dividing polynomials is similar to long division with numbers. Instead of digit by digit, we will proceed term by term.

Example 1. Divide $x^4 + x^3 - 5x^2 + 26x - 21$ by $x^2 + 3x - 4$

Solution: Step 1: Divide the first term by the first term: $\frac{x^4}{x^2} = x^2$

x^2 is the first term of the quotient; we write it above the line.

We multiply x^2 and what we are dividing by: $x^2(x^2 + 3x - 4) = x^4 + 3x^3 - 4x^2$

We take the opposite of the result: $-(x^4 + 3x^3 - 4x^2) = -x^4 - 3x^3 + 4x^2$

We add that to the original polynomial shown above. This is now the new polynomial to divide, and the process begins again.

$$\begin{array}{r}
 x^2 \overline{) } \\
 x^2 + 3x - 4 \phantom{) } \\
 \hline
 x^4 \\
 -x^4 \\
 \hline
 x^3 - 5x^2 + 26x - 21 \\
 -x^4 \\
 \hline
 -2x^3 - x^2 + 26x - 21
 \end{array}$$

Step 2: We now need to divide $-2x^3 - x^2 + 26x - 21$ by $x^2 + 3x - 4$.

Divide the first term by the first term: $\frac{-2x^3}{x^2} = -2x$

$-2x$ is the second term of the quotient; we write it above the line.

We multiply $-2x$ and what we are dividing by:

$$-2x(x^2 + 3x - 4) = -2x^3 - 6x^2 + 8x$$

We take the opposite of the result: $-1(-2x^3 - 6x^2 + 8x) = 2x^3 + 6x^2 - 8x$

We add that to the polynomial to be divided. This sum is now the new polynomial to divide, and the process begins again.

$$\begin{array}{r}
 x^2 \quad + \quad 3x \quad - \quad 4 \quad) \quad \begin{array}{r} x^2 \quad - \quad 2x \\ x^4 \quad + \quad x^3 \quad - \quad 5x^2 \quad + \quad 26x \quad - \quad 21 \\ -x^4 \quad - \quad 3x^3 \quad + \quad 4x^2 \\ \hline - \quad 2x^3 \quad - \quad x^2 \quad + \quad 26x \quad - \quad 21 \\ 2x^3 \quad + \quad 6x^2 \quad - \quad 8x \\ \hline 5x^2 \quad + \quad 18x \quad - \quad 21 \end{array}
 \end{array}$$

Step 3: Divide the first term by the first term: $\frac{5x^2}{x^2} = 5$

5 is the third term of the quotient; we write it above the line.

We multiply 5 and what we are dividing by: $5(x^2 + 3x - 4) = 5x^2 + 15x - 20$

We take the opposite of the result: $-1(5x^2 + 15x - 20) = -5x^2 - 15x + 20$

We add that to the polynomial to be divided.

$$\begin{array}{r}
 x^2 \quad + \quad 3x \quad - \quad 4 \quad) \quad \begin{array}{r} x^2 \quad - \quad 2x \quad + \quad 5 \\ x^4 \quad + \quad x^3 \quad - \quad 5x^2 \quad + \quad 26x \quad - \quad 21 \\ -x^4 \quad - \quad 3x^3 \quad + \quad 4x^2 \\ \hline - \quad 2x^3 \quad - \quad x^2 \quad + \quad 26x \quad - \quad 21 \\ 2x^3 \quad + \quad 6x^2 \quad - \quad 8x \\ \hline 5x^2 \quad + \quad 18x \quad - \quad 21 \\ - \quad 5x^2 \quad - \quad 15x \quad + \quad 20 \\ \hline 3x \quad - \quad 1 \end{array}
 \end{array}$$

For the next step, we would have to divide $3x - 1$ by $x^2 + 3x - 4$. The division of the first terms $\frac{3x}{x^2} = \frac{3}{x}$ is not allowed: this is where we must stop. The quotient is $x^2 - 2x + 5$ and the remainder is $3x - 1$. The result can be presented in two different ways, very much like the division $38 \div 5$ can be represented in two ways:

$$\frac{38}{5} = 7\frac{3}{5} \quad \text{or} \quad 38 \div 5 = 7 \text{ R } 3$$

$$\text{our result is } \frac{x^4 + x^3 - 5x^2 + 26x - 21}{x^2 + 3x - 4} = \boxed{x^2 - 2x + 5 + \frac{3x - 1}{x^2 + 3x - 4}}$$

$$\text{or } (x^4 + x^3 - 5x^2 + 26x - 21) \div (x^2 + 3x - 4) = \boxed{x^2 - 2x + 5 \text{ R } 3x - 1}$$

We can check our result by multiplying back:

$$\begin{aligned}
 P &= (x^2 + 3x - 4) \left(x^2 - 2x + 5 + \frac{3x - 1}{x^2 + 3x - 4} \right) \\
 &= (x^2 + 3x - 4) (x^2 - 2x + 5) + (x^2 + 3x - 4) \frac{3x - 1}{x^2 + 3x - 4} \\
 &= x^2 (x^2 - 2x + 5) + 3x (x^2 - 2x + 5) - 4 (x^2 - 2x + 5) + 3x - 1 \\
 &= x^4 - 2x^3 + 5x^2 + 3x^3 - 6x^2 + 15x - 4x^2 + 8x - 20 + 3x - 1 = x^4 + x^3 - 5x^2 + 26x - 21
 \end{aligned}$$

A special case of division of polynomials is when we divide by a linear polynomial. By the theorem above, such a division results in a remainder that is a polynomial with a degree smaller than linear: that is, a constant polynomial.

Suppose that $N(x)$ is a polynomial and $m(x) = x - a$ is a linear polynomial. According to the theorem above, there exist a polynomial $q(x)$ and real number r such that

$$N(x) = q(x)(x - a) + r$$

The equality above is true between polynomial functions. Thus, the equality above is true for all values of x . Let $x = a$.

$$\begin{aligned} N(x) &= q(x)(x - a) + r && \text{Let } a = x \\ N(a) &= q(a)(a - a) + r \\ N(a) &= q(a) \cdot 0 + r \\ N(a) &= r \end{aligned}$$

Theorem (Remainder Theorem) Suppose that $N(x)$ is a polynomial and $m(x) = x - a$. The remainder of the division of $N(x)$ by $x - a$ is equal to $N(a)$.

Example 2. Use the remainder theorem to find the remainder in the division $(5x^4 - 8x^3 + 3x^2 - x - 1) \div (x - 2)$

Solution: Let $N(x) = 5x^4 - 8x^3 + 3x^2 - x - 1$. If we divide this polynomial by $m(x) = x - 2$, the remainder is the same as $N(a)$. Thus, we evaluate $N(x)$ at $x = 2$.

$$\begin{aligned} N(2) &= 5 \cdot 2^4 - 8 \cdot 2^3 + 3 \cdot 2^2 - 2 - 1 = 5 \cdot 16 - 8 \cdot 8 + 3 \cdot 4 - 2 - 1 \\ &= 80 - 64 + 12 - 2 - 1 = \boxed{25} \end{aligned}$$

Thus the remainder is 25.

An immediate consequence of the remainder theorem is when this remainder is zero.

Theorem: Suppose that $N(x)$ is a polynomial and a is a real number such that $N(a) = 0$. Then $x - a$ is a factor of $N(x)$, i.e. there exists a polynomial $q(x)$ such that $N(x) = (x - a)q(x)$

This is a very important result. This theorem proves that every zero of every polynomial is caused by linear factors. So, for example, if a polynomial $P(x)$ is so that $P(4) = 0$, then $P(x)$ can be factored, and one of the factors is $(x - 4)$. In this sense, this theorem is another factoring technique. Whenever we find a zero, we also find a linear factor.

Example 3. Solve the equation $x^7 - 17x^5 + 36x^4 - 20x^3 = 0$ given that 2 is a solution of the equation.

Solution: To solve any equation of degree greater than one, we need to factor and apply the zero product rule. The greatest common factor is x^3 . We factor it out:

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3(x^4 - 17x^2 + 36x - 20)$$

We still need to factor $P(x) = x^4 - 17x^2 + 36x - 20$. Since $P(2) = 0$, $x - 2$ is a linear factor of $P(x)$. We divide $x^4 - 17x^2 + 36x - 20$ by $x - 2$. (For the steps to dividing a polynomials, please see the separate handout on dividing polynomials.) The quotient is $x^3 + 2x^2 - 13x + 10$. So far, we have that

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3(x - 2)(x^3 + 2x^2 - 13x + 10)$$

We still need to factor $Q(x) = x^3 + 2x^2 - 13x + 10$. Let us substitute $x = 2$ into $Q(x)$.

$$Q(2) = 2^3 + 2 \cdot 2^2 - 13 \cdot 2 + 10 = 8 + 8 - 26 + 10 = 0$$

This means that $x - 2$ is a factor of $Q(x)$. We divide: $x^3 + 2x^2 - 13x + 10$ by $x - 2$ and obtain the quotient $x^2 + 4x - 5$ and the remainder is zero. So now we have

$$x^7 - 17x^5 + 36x^4 - 20x^3 = x^3(x - 2)(x - 2)(x^2 + 4x - 5) = x^3(x - 2)^2(x^2 + 4x - 5)$$

We factor the quadratic expression $x^2 + 4x - 5$ and obtain $(x + 5)(x - 1)$. Thus, we now have completely factored our polynomial:

$$\begin{aligned} x^7 - 17x^5 + 36x^4 - 20x^3 &= 0 \\ x^3(x - 2)^2(x + 5)(x - 1) &= 0 \end{aligned}$$

Applying the zero product rule, we obtain the solution set $\{-5, 0, 1, 2\}$.

Another important application of division of polynomials is when we graph rational functions.

Example 4. Graph $f(x) = \frac{3x + 5}{x + 2}$

Solution: $f(x) = \frac{3x + 5}{x + 2}$ is a rational function where both numerator and denominator are linear. If that's the case, the graph of the function is a "relative" of that of $y = \frac{1}{x}$. We will divide the polynomials and bring the function to a form that helps us see what transformations are needed and in what order. $(3x + 5)$ divided by $(x + 2)$ is 3 with a remainder of -1 . We can use long division, or smuggling:

$$\frac{3x + 5}{x + 2} = \frac{3x + 6 - 6 + 5}{x + 2} = \frac{3x + 6}{x + 2} - \frac{1}{x + 2} = 3 - \frac{1}{x + 2} = -1 \cdot \frac{1}{x + 2} + 3$$

This means that we can graph f using transformations as follows.

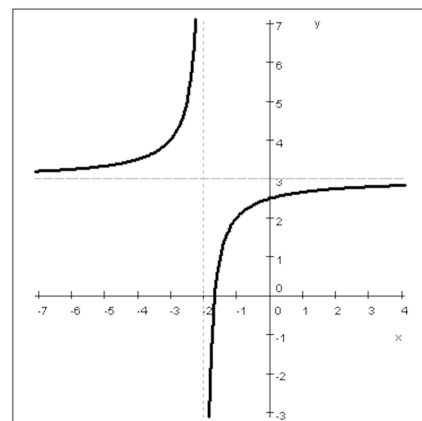
Step 1. We graph $y = \frac{1}{x}$.

Step 2. We shift the graph to the left by 2 units

Step 3. We reflect the graph to the x -axis.

Step 4. We shift the graph upward by 3 units.

Note that the horizontal and vertical asymptotes are not part of the graph, but they are very helpful in performing the transformations on the graph.





Practice Problems

1. Perform the indicated divisions with remainders:

- a) $132 \div 7$ b) $1145 \div 12$ c) $918 \div 8$ d) $201 \div 12$

2. Perform each of the following divisions with remainder.

- a) $(3x^5 - 8x^4 + 11x^2 + 2x - 7) \div (x^2 + x)$ g) $(x^6 - 1) \div (x - 1)$
 b) $(3x^5 - 8x^4 + 11x^2 + 2x - 7) \div (x^3 - 3x)$ h) $(x^6 - 1) \div (x + 2)$
 c) $(3x^5 - 8x^4 + 11x^2 + 2x - 7) \div (x^2 + 1)$ i) $(x^6 - 1) \div (x^2 - 4)$
 d) $(5x^5 - 24x^3 - 7x^2 - 5x + 35) \div (x - 1)$ j) $(x^2 - 7x + 16) \div (x - 3)$
 e) $(5x^5 - 24x^3 - 7x^2 - 5x + 35) \div (x^3 + 2x - 1)$ k) $(x^5 - 2) \div (x - 2)$
 f) $(5x^5 - 24x^3 - 7x^2 - 5x + 35) \div (x^2 - 5)$

3. Use the remainder theorem to find the remainder in each of the following divisions.

- a) $(8x^5 - 2x^3 + x + 11) \div (x - 1)$ c) $(x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2) \div (x - 2)$
 b) $(8x^5 - 2x^3 + x + 11) \div (x + 1)$ d) $(x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2) \div (x + 2)$

4. Solve each of the given equations.

- a) $x^4 + 7x^3 - 4x^2 - 28x = 0$ given that -2 is a solution
 b) $x^6 + x^5 - 18x^4 - 52x^3 - 40x^2 = 0$ given that 2 is a solution
 c) $x^4 + 8x^3 + 12x^2 - 32x - 64 = 0$ given that -4 is a solution

5. Solve each of the following equations.

- a) $x^6 - 15x^5 + 53x^4 - 21x^3 - 90x^2 = 0$
 b) $x^6 + 7x^5 - 47x^4 - 307x^3 - 394x^2 + 236x + 504 = 0$
 c) $x^6 - 2x^5 - 50x^4 + 4x^3 + 97x^2 - 2x - 48 = 0$

6. Find all the values of k such that $f(x) = k^2x^3 - 7kx + 12$ is divisible by $(x - 1)$.

7. Graph each of the given functions.

- a) $f(x) = \frac{4x - 2}{x - 1}$ b) $f(x) = \frac{3x + 13}{x + 4}$ c) $f(x) = \frac{2x - 7}{x - 3}$



Answers

Practice Problems

1. a) 18 R 6 b) 95 R 5 c) 114 R 6 d) 16 R 9 2. a) 18 b) 4 c) 84 d) 0
3. a) $3x^3 - 11x^2 + 11x$ R $2x - 7$ b) $3x^2 - 8x + 9$ R $-13x^2 + 29x - 7$ c) $3x^3 - 8x^2 - 3x + 19$ R $5x - 26$
 d) $5x^4 + 5x^3 - 19x^2 - 26x - 31$ R 4 e) $5x^2 - 34$ R $-2x^2 + 63x + 1$ f) $5x^3 + x - 7$
 g) $x^5 + x^4 + x^3 + x^2 + x + 1$ h) $x^5 - 2x^4 + 4x^3 - 8x^2 + 16x - 32$ R 63 i) $x^4 + 4x^2 + 16$ R 63
 j) $x - 4$ R 4 k) $x^4 + 2x^3 + 4x^2 + 8x + 16$ R 30
4. a) $-7, -2, 0, 2$ b) $-2, 0, 5$ c) $-4, -2, 2$ 5. a) $-1, 0, 3, 10$ b) $-9, -2, 1, 7$ c) $-6, -1, 1, 8$ 6. 3, 4
7. a) $f(x) = \frac{4x-2}{x-1} = \frac{2}{x-1} + 4$ b) $f(x) = \frac{3x+13}{x+4} = \frac{1}{x+4} + 3$ c) $f(x) = \frac{2x-7}{x-3} = 2 - \frac{1}{x-3}$

