

Sample Problems

Solve each of the non-linear systems of equations.

$$1. \begin{cases} y = x^2 + 6x - 10 \\ y + 5 = 2x \end{cases}$$

$$4. \begin{cases} (x-4)^2 + (y+1)^2 = 20 \\ x + 3y = 11 \end{cases}$$

$$7. \begin{cases} (x+1)^2 + (y-2)^2 = 25 \\ 4y = 3x + 11 \end{cases}$$

$$2. \begin{cases} y = 2x^2 - 8x + 1 \\ y = -4x - 1 \end{cases}$$

$$5. \begin{cases} (x+5)^2 + (y-6)^2 = 10 \\ y = -\frac{1}{3}x + 1 \end{cases}$$

$$8. \begin{cases} x + y = -1 \\ xy = -12 \end{cases}$$

$$3. \begin{cases} y = -x^2 + 3x - 5 \\ y = x + 1 \end{cases}$$

$$6. \begin{cases} (x-3)^2 + (y-1)^2 = 20 \\ x - y = -5 \end{cases}$$

Practice Problems

Solve each of the non-linear systems of equations.

$$1. \begin{cases} y = x^2 + 4x - 18 \\ y + 3 = 2x \end{cases}$$

$$7. \begin{cases} x^2 + (y+4)^2 = 20 \\ x - 3y = 2 \end{cases}$$

$$13. \begin{cases} (x-1)^2 + (y+7)^2 = 25 \\ y = -\frac{1}{2}x - \frac{3}{2} \end{cases}$$

$$2. \begin{cases} y = x^2 + 4x - 12 \\ y = -2x - 21 \end{cases}$$

$$8. \begin{cases} (x-2)^2 + (y+5)^2 = 10 \\ 3y - x = -7 \end{cases}$$

$$14. \begin{cases} (x-2)^2 + (y-1)^2 = 25 \\ 3y = 4x - 30 \end{cases}$$

$$3. \begin{cases} y = -x^2 + 3x + 1 \\ y = 7x + 9 \end{cases}$$

$$9. \begin{cases} (x-3)^2 + (y+1)^2 = 16 \\ y + x = 12 \end{cases}$$

$$15. \begin{cases} (x+1)^2 + (y-4)^2 = 17 \\ x + 4y = 15 \end{cases}$$

$$4. \begin{cases} y = x^2 - 5x - 4 \\ y = x - 12 \end{cases}$$

$$10. \begin{cases} (x-4)^2 + (y-3)^2 = 25 \\ 3x + 4y = 24 \end{cases}$$

$$16. \begin{cases} x + y = 12 \\ xy = 35 \end{cases}$$

$$5. \begin{cases} y = -\frac{1}{2}x^2 + 3x + 4 \\ y = 5x + 6 \end{cases}$$

$$11. \begin{cases} (x-2)^2 + (y+1)^2 = 20 \\ x - 2y = 19 \end{cases}$$

$$17. \begin{cases} x^2 + y^2 = 68 \\ x + y = -6 \end{cases}$$

$$6. \begin{cases} y = -x^2 - 6x + 1 \\ y = -2x + 9 \end{cases}$$

$$12. \begin{cases} (x+1)^2 + (y-3)^2 = 50 \\ x - 7y + 72 = 0 \end{cases}$$

$$18. \begin{cases} x - y = 4 \\ \frac{1}{x} + \frac{1}{y} = \frac{2}{3} \end{cases}$$

Sample Problems - Answers

- 1.) $(-5, -15)$ and $(1, -3)$ 2.) $(1, -5)$ 3.) no solution 4.) $(8, 1)$ and $(2, 3)$ 5.) $(-6, 3)$
- 6.) no solution 7.) $(3, 5)$ and $(-5, -1)$ 8.) $(3, -4)$ and $(-4, 3)$

Practice Problems - Answers

- 1.) $(3, 3)$ and $(-5, -13)$ 2.) $(-3, -15)$ 3.) no solution 4.) $(2, -10)$ and $(4, -8)$ 5.) $(-2, -4)$
- 6.) no solution 7.) $(4, -2)$ and $(2, 0)$ 8.) $(1, -2)$ 9.) no solution 10.) $(0, 6)$ and $(8, 0)$
- 11.) no solution 12.) $(-2, 10)$ 13.) $(1, -2)$ and $(5, -4)$ 14.) $(6, -2)$ 15.) $(3, 3)$ and $(-5, 5)$
- 16.) $(5, 7)$ and $(7, 5)$ 17.) $(-8, 2)$ and $(2, -8)$ 18.) $(6, 2)$ and $(1, -3)$

Sample Problems - Solutions

Solve each of the non-linear systems of equations.

$$1. \begin{cases} y = x^2 + 6x - 10 \\ y + 5 = 2x \end{cases}$$

Solution: We will use substitution. We solve for y in the second equation: $y = 2x - 5$. We substitute this into the first equation and solve for x .

$$\begin{aligned} 2x - 5 &= x^2 + 6x - 10 \\ 0 &= x^2 + 4x - 5 \\ 0 &= (x + 5)(x - 1) \implies x_1 = -5 \text{ and } x_2 = 1 \end{aligned}$$

We can find the value of y by using either of the equations, so let us use the easier one which is linear: $y = 2x - 5$.

$$y_1 = 2x_1 - 5 = 2(-5) - 5 = -15 \text{ and } y_2 = 2x_2 - 5 = 2(1) - 5 = -3$$

Thus the solutions are $x_1 = -5$ and $y_1 = -15$ or $x_2 = 1$ and $y_2 = -3$. Or, using a more compact notation, the solutions are $(-5, -15)$ and $(1, -3)$.

We check: for the solution $(-5, -15)$

$$\begin{aligned} y &= x^2 + 6x - 10 : \quad \text{RHS} = x^2 + 6x - 10 = (-5)^2 + 6(-5) - 10 = 25 - 30 - 10 = -15 = \text{LHS} \\ y &= 2x - 5 : \quad \text{RHS} = 2(-5) - 5 = -15 = \text{LHS} \end{aligned}$$

Thus $(-5, -15)$ is indeed a solution. We also check $(1, -3)$

$$\begin{aligned} y &= x^2 + 6x - 10 : \quad \text{RHS} = x^2 + 6x - 10 = (1)^2 + 6 \cdot 1 - 10 = 1 + 6 - 10 = -3 = \text{LHS} \\ y &= 2x - 5 : \quad \text{RHS} = 2(1) - 5 = -3 = \text{LHS} \end{aligned}$$

Thus $(1, -3)$ is also a solution, and so both solutions are correct.

$$2. \begin{cases} y = 2x^2 - 8x + 1 \\ y = -4x - 1 \end{cases}$$

Solution: We will use substitution. We substitute $y = -4x - 1$ into the first equation and solve for x .

$$\begin{aligned} -4x - 1 &= 2x^2 - 8x + 1 \\ 0 &= 2x^2 - 4x + 2 \\ 0 &= 2(x^2 - 2x + 1) \\ 0 &= 2(x - 1)^2 \implies x = 1 \end{aligned}$$

There is one solution for x . We can find the value of y by using either of the equations, so let us use the simpler one, $y = -4x - 1 = -4(1) - 1 = -5$. Thus the solution is $(1, -5)$. We check: $(-1, 5)$ must be the solution of both equations.

$$\begin{aligned} y &= 2x^2 - 8x + 1 : \quad \text{RHS} = 2(1)^2 - 8 \cdot 1 + 1 = 2 - 8 + 1 = -5 = \text{LHS} \\ y &= -4x - 1 : \quad \text{RHS} = -4 \cdot 1 - 1 = -5 = \text{LHS} \end{aligned}$$

Thus $(1, -5)$ is indeed a solution.

$$3. \begin{cases} y = -x^2 + 3x - 5 \\ y = x + 1 \end{cases}$$

Solution: We will use substitution. We substitute $y = x + 1$ into the first equation and solve for x .

$$\begin{aligned} x + 1 &= -x^2 + 3x - 5 \\ 0 &= -x^2 + 2x - 6 \\ 0 &= -(x^2 - 2x + 6) \\ 0 &= -[(x - 1)^2 + 5] \implies \text{no real solution for } x \end{aligned}$$

The fact that the equation has no real solution for x indicates that the system has no solution among the pairs of real numbers.

$$4. \begin{cases} (x - 4)^2 + (y + 1)^2 = 20 \\ x + 3y = 11 \end{cases}$$

Solution: We will use substitution. We solve for x in the second equation: $x = -3y + 11$. We substitute this into the first equation and solve for y .

$$\begin{aligned} (x - 4)^2 + (y + 1)^2 &= 20 & 10y^2 - 40y + 30 &= 0 \\ (-3y + 11 - 4)^2 + (y + 1)^2 &= 20 & 10(y^2 - 4y + 3) &= 0 \\ (-3y + 7)^2 + (y + 1)^2 &= 20 & 10(y - 1)(y - 3) &= 0 \\ 9y^2 - 42y + 49 + y^2 + 2y + 1 &= 20 & y_1 = 1 \text{ and } y_2 = 3 & \\ 10y^2 - 40y + 50 &= 20 & & \end{aligned}$$

There are two solutions for y . For each, we can find the value of x by using the equation $x = -3y + 11$. At this point, it is important to use the linear equation because if we used the quadratic equation, we would end up with 4 solutions, 2 correct, 2 incorrect. So, we proceed to use the linear equation to find x .

$$x_1 = -3y_1 + 11 = -3(1) + 11 = 8 \text{ and } x_2 = -3y_2 + 11 = -3(3) + 11 = 2$$

Thus the solutions are $(8, 1)$ and $(2, 3)$.

We check: the coordinates of an intersection point must be the solution of both equations. For the $(8, 1)$, we check

$$\begin{aligned} (x - 4)^2 + (y + 1)^2 &= 20 : & \text{LHS} &= (8 - 4)^2 + (1 + 1)^2 = 4^2 + 2^2 = 16 + 4 = 20 = \text{RHS} \\ x + 3y &= 11 : & \text{LHS} &= 8 + 3 \cdot 1 = 11 = \text{RHS} \end{aligned}$$

Thus $(8, 1)$ is indeed a solution. We also check $(2, 3)$

$$\begin{aligned} (x - 4)^2 + (y + 1)^2 &= 20 : & \text{LHS} &= (2 - 4)^2 + (3 + 1)^2 = (-2)^2 + 4^2 = 4 + 16 = 20 = \text{RHS} \\ x + 3y &= 11 : & \text{LHS} &= 2 + 3 \cdot 3 = 11 = \text{RHS} \end{aligned}$$

Thus $(2, 3)$ is also a solution.

Please note that our choice to solve for x and substitute that was wise. If instead we solved for y , we would have to use the substitution $y = \frac{-x + 11}{3}$ and thus solve the quadratic equation

$$(x - 4)^2 + \left(\frac{-x + 11}{3} + 1 \right)^2 = 20 \text{ which is much more laborious than what we did.}$$

$$5. \begin{cases} (x+5)^2 + (y-6)^2 = 10 \\ y = -\frac{1}{3}x + 1 \end{cases}$$

Solution: We will use substitution. We solve for x in the second equation:

$$\begin{aligned} y &= -\frac{1}{3}x + 1 & 3y + x &= 3 \\ 3y &= -x + 3 & x &= -3y + 3 \end{aligned}$$

We substitute $x = -3y + 3$ into the first equation and solve for y .

$$\begin{aligned} (x+5)^2 + (y-6)^2 &= 10 & 10y^2 - 60y + 90 &= 0 \\ (-3y+3+5)^2 + (y-6)^2 &= 10 & 10(y^2 - 6y + 9) &= 0 \\ (-3y+8)^2 + (y-6)^2 &= 10 & 10(y-3)^2 &= 0 \\ 9y^2 - 48y + 64 + y^2 - 12y + 36 &= 10 & y &= 3 \\ 10y^2 - 60y + 100 &= 10 \end{aligned}$$

We can find the x -value by using the equation $x = -3y + 3$.

$$x = -3y + 3 = -3(3) + 3 = -9 + 3 = -6$$

Thus the solution is $(-6, 3)$. We check:

$$\begin{aligned} (x+5)^2 + (y-6)^2 &= 10 : & \text{LHS} &= (-6+5)^2 + (3-6)^2 = (-1)^2 + (-3)^2 = 1 + 9 = 10 = \text{RHS} \\ y &= -\frac{1}{3}x + 1 : & \text{RHS} &= -\frac{1}{3}(-6) + 1 = 2 + 1 = 3 = \text{LHS} \end{aligned}$$

Thus $(-6, 3)$ is indeed a solution.

$$6. \begin{cases} (x-3)^2 + (y-1)^2 = 20 \\ x - y = -5 \end{cases}$$

Solution: We will use substitution. We solve for y in the second equation: $y = x + 5$. We substitute this into the first equation and solve for y .

$$\begin{aligned} (x-3)^2 + (y-1)^2 &= 20 & 2x^2 + 2x + 25 &= 20 \\ (x-3)^2 + (x+5-1)^2 &= 20 & 2x^2 + 2x + 5 &= 0 \\ (x-3)^2 + (x+4)^2 &= 20 & 2\left(x^2 + x + \frac{5}{2}\right) &= 0 \\ x^2 - 6x + 9 + x^2 + 8x + 16 &= 20 & 2\left(\left(x + \frac{1}{2}\right)^2 + \frac{9}{4}\right) &= 0 \end{aligned}$$

There is no real solution of this equation. This indicates that there is no real solution for this system.

$$7. \begin{cases} (x+1)^2 + (y-2)^2 = 25 \\ 4y = 3x + 11 \end{cases}$$

Solution: We will use substitution. This problem is different from the previous problems because we can not solve for x or y without division, and so the algebra will be slightly more laborous. There are a few tricks to simplify computation in such a situation. We solve for y in the second equation: $y = \frac{3x+11}{4}$. We substitute this into the first equation and solve for y .

$$\begin{aligned} (x+1)^2 + (y-2)^2 &= 25 & (x+1)^2 + \frac{9}{16}(x+1)^2 &= 25 \\ (x+1)^2 + \left(\frac{3x+11}{4} - 2\right)^2 &= 25 & \frac{25}{16}(x+1)^2 &= 25 \\ (x+1)^2 + \left(\frac{3x+11}{4} - \frac{8}{4}\right)^2 &= 25 & (x+1)^2 &= 16 \\ (x+1)^2 + \left(\frac{3x+3}{4}\right)^2 &= 25 & (x+1)^2 - 16 &= 0 \\ (x+1)^2 + \frac{(3x+3)^2}{4^2} &= 25 & (x+1+4)(x+1-4) &= 0 \\ (x+1)^2 + \frac{[3(x+1)]^2}{16} &= 25 & (x+5)(x-3) &= 0 \\ (x+1)^2 + \frac{9(x+1)^2}{16} &= 25 & x_1 = -5 \text{ and } x_2 = 3 & \end{aligned}$$

We can find the x -coordinates using the equation

$$y = \frac{3x+11}{4}.$$

$$y_1 = \frac{3x_1+11}{4} = \frac{3(-5)+11}{4} = -1 \text{ and } y_2 = \frac{3x_2+11}{4} = \frac{3(3)+11}{4} = 5$$

Thus the solutions are $(-5, -1)$ and $(3, 5)$.

$$8. \begin{cases} x+y = -1 \\ xy = -12 \end{cases}$$

Solution: We will use substitution. From the first equation, $y = -x - 1$. Then the second equation becomes

$$\begin{aligned} xy &= -12 \\ x(-x-1) &= -12 \\ -x^2 - x &= -12 \\ 0 &= x^2 + x - 12 \\ 0 &= (x+4)(x-3) \implies x_1 = -4 \quad x_2 = 3 \end{aligned}$$

If $x_1 = -4$, then $y_1 = -x_1 - 1 = -(-4) - 1 = 4 - 1 = 3$. If $x_2 = 3$, then $y_2 = -x_2 - 1 = -3 - 1 = -4$. Thus the system has two solutions: $(3, -4)$ and $(-4, 3)$.