

We will now further study the quadratic formula and its properties. Let us first recall the formula.

Theorem: Suppose that a , b , and c are real numbers, with $a \neq 0$. Then the solutions of the equation

$$ax^2 + bx + c = 0 \text{ are}$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions of such a polynomial equations are also called zeroes or roots. The zeroes, as always, are conjugates of each other. We remember that conjugates have nice algebraic properties: there is cancellation whenever we add, subtract, or multiply them. Because of that, even if the roots themselves are irrational, their sum and product are surprisingly simple and closely related to the coefficients a , b , and c .

Let s_1 and s_2 be the two solutions of the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$). Let us look at the sum of the two solutions.

$$s_1 + s_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

The product is also surprisingly simple. When we multiply conjugates, O and I in FOIL cancel out each other, and we are left with the difference of two squares.

$$\begin{aligned} s_1 s_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2} \\ &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} \end{aligned}$$

These properties are sometimes called the Vieta's formulas after French mathematician François Viète.

Theorem: (Vieta's formulas) Suppose that a , b , and c are real numbers, with $a \neq 0$, and s_1 and s_2 are the solutions of the equation $ax^2 + bx + c = 0$. If the solutions exist, then

$$s_1 + s_2 = -\frac{b}{a} \quad \text{and} \quad s_1 s_2 = \frac{c}{a}$$

Example 1. Find the sum and product of the solutions of the given equation. $3x^2 + 7x - 1 = 0$

Solution: We can apply Vieta's formulas with coefficients $a = 3$, $b = 7$, and $c = -1$.

$$s_1 + s_2 = -\frac{b}{a} = -\frac{7}{3} \quad \text{and} \quad s_1 s_2 = \frac{c}{a} = -\frac{1}{3}$$

No matter how irrational the solutions themselves are, these values are always rational, as long as a , b , and c are rational. But there is one catch. We will get results for the product and sum of the roots even if the real solutions don't exist. We have to check for that. In the previous case, the discriminant is positive, $b^2 - 4ac = 7^2 - 4 \cdot 3 \cdot (-1) = 61$ and so the solutions do exist.

Example 2. Find the sum and product of the real solutions of the given equation. $2x^2 - x + 5 = 0$

Solution: We could easily get the values for the sum and product, but in this case, the solutions don't exist among the real numbers. Let us compute the discriminant. $b^2 - 4ac = (-1)^2 - 4 \cdot 2 \cdot 5 = -19$. Therefore, there is no value for these values.

Given the sum and the product, the value of many other expressions can be obtained.

Example 3. Let s_1 and s_2 be the solutions of the equation. $6x^2 - x - 3 = 0$. Find the exact value of each of the given expressions.

a) $s_1^2 + s_2^2$ b) $\frac{1}{s_1} + \frac{1}{s_2}$ c) $s_1^3 + s_2^3$

Solution: a) The first instinct might be to take the sum of the two roots and square. However, that is not exactly what we want: $(s_1 + s_2)^2 = s_1^2 + s_2^2 + 2s_1s_2$

We can solve for $s_1^2 + s_2^2$ in this equation and get $s_1^2 + s_2^2 = (s_1 + s_2)^2 - 2s_1s_2$. This can be expressed using Vieta's formulas.

$$s_1^2 + s_2^2 = (s_1 + s_2)^2 - 2s_1s_2 = \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{1}{36} - \frac{2(-3)}{6} = \frac{37}{36}$$

b) If we start with the expression given and bring the fractions to the common denominator, we quickly obtain an expression in terms of the the sum and product of the roots.

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{s_1 + s_2}{s_1s_2} = \frac{\left(\frac{-b}{a}\right)}{\left(\frac{c}{a}\right)} = -\frac{b}{c} = -\frac{(-1)}{-3} = -\frac{1}{3}$$

c) Similar to the methods in part a), we can start with $(s_1 + s_2)^3$ and subtract what is extra.

$$\begin{aligned}(s_1 + s_2)^3 &= (s_1 + s_2)(s_1 + s_2)^2 = (s_1 + s_2)(s_1^2 + 2s_1s_2 + s_2^2) \\ &= s_1^3 + 2s_1^2s_2 + s_1s_2^2 + s_1^2s_2 + 2s_1s_2^2 + s_2^3 = s_1^3 + s_2^3 + 3s_1s_2^2 + 3s_1^2s_2\end{aligned}$$

Now we have the equation $(s_1 + s_2)^3 = s_1^3 + s_2^3 + 3s_1s_2^2 + 3s_1^2s_2$. We can solve for $s_1^3 + s_2^3$ here:

$$\begin{aligned}s_1^3 + s_2^3 &= (s_1 + s_2)^3 - 3s_1s_2^2 - 3s_1^2s_2 = (s_1 + s_2)^3 - 3s_1s_2(s_1 + s_2) \text{ factor out } s_1 + s_2 \\ &= (s_1 + s_2)\left((s_1 + s_2)^2 - 3s_1s_2\right)\end{aligned}$$

We can now use Vieta's formulas:

$$\begin{aligned}s_1^3 + s_2^3 &= (s_1 + s_2)\left((s_1 + s_2)^2 - 3s_1s_2\right) = \left(\frac{-b}{a}\right)\left(\left(\frac{-b}{a}\right)^2 - 3 \cdot \frac{c}{a}\right) \\ &= -\frac{(-1)}{6}\left(\left(\frac{-(-1)}{6}\right)^2 - 3 \cdot \frac{-3}{6}\right) = \frac{1}{6}\left(\frac{1}{36} + \frac{9}{6}\right) = \frac{1}{6} \cdot \frac{1 + 54}{36} = \frac{55}{216}\end{aligned}$$

Example 4. Find the value of c in the equation $5x^2 + 10x + c = 0$ if we know that $s_1^2 + s_2^2 = \frac{36}{5}$, where s_1 and s_2 are the (real) solutions of the equation.

Solution: We have expressed $s_1^2 + s_2^2$ using Vieta's formulas. We will now apply them to obtain an equation for a .

$$s_1^2 + s_2^2 = (s_1 + s_2)^2 - 2s_1s_2 = \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \left(\frac{-10}{5}\right)^2 - \frac{2c}{5} = 4 - \frac{2c}{5}$$

Therefore, we now have the equation for $s_1^2 + s_2^2$:

$$\begin{array}{rcl} 4 - \frac{2c}{5} & = & \frac{36}{5} & \text{divide by 2} \\ 2 - \frac{c}{5} & = & \frac{18}{5} & \text{multiply by 5} \\ 10 - c & = & 18 \\ -8 & = & c \end{array}$$

Thus, $c = -8$. We must however, make sure that the solutions exist when $c = -8$. Computing the discriminant for $a = 5$, $b = 10$, and $c = -8$,

$$b^2 - 4ac = 100 - 4 \cdot 5 \cdot (-8) = 260 > 0$$

Since the discriminant is positive, there are two solutions for the equations when $c = -8$.

Example 5. Find the value of a in the equation $ax^2 - x - 2 = 0$ if we know that $s_1^2 + s_2^2 = 21$, where s_1 and s_2 are the (real) solutions of the equation.

Solution: We have expressed $s_1^2 + s_2^2$ using Vieta's formulas. We will now apply them to obtain an equation for a .

$$s_1^2 + s_2^2 = (s_1 + s_2)^2 - 2s_1s_2 = \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \left(\frac{1}{a}\right)^2 - \frac{2(-2)}{a} = \frac{1}{a^2} + \frac{4}{a}$$

Therefore, we now have the equation for $s_1^2 + s_2^2$:

$$\begin{array}{rcl} \frac{1}{a^2} + \frac{4}{a} & = & 21 & \text{multiply by } a^2 \\ 1 + 4a & = & 21a^2 \\ 21a^2 - 4a - 1 & = & 0 \end{array}$$

$$a_{1,2} = \frac{4 \pm \sqrt{16 - 4(-1)21}}{42} = \frac{4 \pm \sqrt{100}}{42} = \frac{4 \pm 10}{42} = \frac{2 \pm 5}{21} = \begin{cases} \frac{2+5}{21} = \frac{7}{21} = \frac{1}{3} \\ \frac{2-5}{21} = \frac{-3}{21} = -\frac{1}{7} \end{cases}$$

If $a = \frac{1}{3}$, then the discriminant is $b^2 - 4ac = (-1)^2 - 4 \cdot \frac{1}{3} \cdot (-2) = 1 + \frac{8}{3} = \frac{11}{3}$. So $a = \frac{1}{3}$ is a solution. If $a = -\frac{1}{7}$, then

the discriminant is $b^2 - 4ac = (-1)^2 - 4 \left(-\frac{1}{7}\right) (-2) = 1 - \frac{8}{7} = -\frac{1}{7}$. So $a = -\frac{1}{7}$ does not work, because if $a = -\frac{1}{7}$,

the discriminant is negative, indicating that the equation $-\frac{1}{7}x^2 - x - 2 = 0$ has no real solution. So $a = \frac{1}{3}$ is the only solution.

Example 6. Find the value of p in the equation $px^2 - px + 2x - 3p + 4 = 0$ if we know that $\frac{1}{s_1} + \frac{1}{s_2} = -\frac{3}{11}$, where s_1 and s_2 are the (real) solutions of the equation.

Solution: We have expressed $\frac{1}{s_1} + \frac{1}{s_2}$ using Vieta's formulas. We will now apply them to obtain an equation for p .

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{s_1 + s_2}{s_1 s_2} = \frac{\left(-\frac{b}{a}\right)}{\frac{c}{a}} = -\frac{b}{c}$$

In the equation given, we have that $px^2 + (-p + 2)x + (-3p + 4) = 0$. Therefore, the coefficients are $a = p$, $b = p - 2$, and $c = -3p + 4$. So the equation we write for the value of $\frac{1}{s_1} + \frac{1}{s_2}$ is:

$$\begin{aligned} -\frac{-p+2}{-3p+4} &= -\frac{3}{11} && \text{multiply by } -1 \\ \frac{-p+2}{-3p+4} &= \frac{3}{11} && \text{multiply by } 11(-3p+4) \\ 11(-p+2) &= 3(-3p+4) \\ -11p+22 &= -9p+12 \\ 10 &= 2p \\ p &= 5 \end{aligned}$$

It appears that $p = 5$ is the solution. We need to check however if the solutions exist when $p = 5$. The equation $px^2 - px + 2x - 3p + 4 = 0$ becomes $5x^2 - 5x + 2x - 15 + 4 = 0$ and $5x^2 - 3x - 11 = 0$. In this case, the discriminant is $b^2 - 4ac = (-3)^2 - 4(5)(-11) = 229$, which is positive. Therefore, $p = 5$ is indeed a solution.



Practice Problems

- Consider the equation $5x^2 - 4x - 3 = 0$. Suppose the solutions of this equation are s_1 and s_2 . Find each of the following, without solving the equation.

a) $s_1 + s_2$	b) $s_1 s_2$	c) $s_1^2 + s_2^2$	d) $\frac{1}{s_1} + \frac{1}{s_2}$	e) $s_1^4 + s_2^4$	f) $s_1^3 + s_2^3$
----------------	--------------	--------------------	------------------------------------	--------------------	--------------------
- Consider the equation $3x^2 + 2x - 7 = 0$. Suppose the solutions of this equation are s_1 and s_2 . Find each of the following, without solving the equation.

a) $s_1 + s_2$	b) $s_1 s_2$	c) $s_1^2 + s_2^2$	d) $\frac{1}{s_1} + \frac{1}{s_2}$	e) $s_1^4 + s_2^4$	f) $s_1^3 + s_2^3$
----------------	--------------	--------------------	------------------------------------	--------------------	--------------------
- Find the value of c in the equation $2x^2 - 5x + c = 0$ if we know that $s_1^2 + s_2^2 = \frac{33}{4}$, where s_1 and s_2 are the (real) solutions of the equation.
- Find the value of b in the equation $3x^2 + bx - 1 = 0$ if we know that $s_1^2 + s_2^2 = \frac{70}{9}$, where s_1 and s_2 are the (real) solutions of the equation.
- Find the value of p if $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{5}$ where s_1 and s_2 are the (real) solutions of the equation $3x^2 + px + 3x + 3p + 1 = 0$



Answers

1. a) $\frac{4}{5}$ b) $-\frac{3}{5}$ c) $\frac{46}{25}$ d) $-\frac{4}{3}$ e) $\frac{1666}{625}$ f) $\frac{244}{125}$
2. a) $-\frac{2}{3}$ b) $-\frac{7}{3}$ c) $\frac{46}{9}$ d) $\frac{2}{7}$ e) $\frac{1234}{81}$ f) $-\frac{134}{27}$
3. -2 4. $-8, 8$ 5. -2