

Translated from the work of Zoltan Tuzson, teacher in Szekelyudvarhely.

Notation: a , b , and c will denote the sides of a triangle, A the area, P the perimeter, (half of the perimeter), R the radius of circumscribed circle, r the radius of the inscribed circle, α , β , γ the angles opposite to side a , b , and c , correspondingly, h_a , h_b , and h_c the height belonging to side a , b , and c , correspondingly. Let m_a , m_b , and m_c denote the medians belonging to each side. (A median is the line segment connecting the midpoint of a side with its opposite vertex.) Let i_α , i_β , and i_γ denote the length of the angle bisectors (the line segment that intersects the triangle).

Prove each of the following.

1. In any triangle, $A = \frac{1}{2}ab \sin \gamma$.
2. (the law of sines) In any triangle, $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$.
3. In any triangle, $R = \frac{abc}{4A}$.
4. In any triangle, $a = b \cos \gamma + c \cos \beta$.
5. In any triangle, $b \cos \beta + c \cos \gamma = a \cos (\beta - \gamma)$.
6. (the law of cosines) In any triangle, $c^2 = a^2 + b^2 - 2ab \cos \gamma$.
7. In any triangle, $b \cos \gamma - c \cos \beta = \frac{b^2 - c^2}{a}$.
8. In any triangle, $A = \frac{a^2 \sin \beta \sin \gamma}{2 \sin \alpha}$.
9. If $a \sin \beta + b \sin \alpha = 2c \sin \gamma$ and $b \sin \gamma + c \sin \beta = 2a \sin \alpha$, then the triangle is equilateral.
10. In any triangle, $\sqrt{a \sin \alpha} + \sqrt{b \sin \beta} + \sqrt{c \sin \gamma} = \sqrt{P(\sin \alpha + \sin \beta + \sin \gamma)}$.
11. In any triangle, $\sin \alpha \cdot \sin \beta \cdot \sin \gamma = \frac{A}{2R^2}$.
12. In any triangle, $\sin \alpha + \sin \beta + \sin \gamma = \frac{P}{2R}$.
13. If $b + c = 3a$, then $\sin \beta + \sin \gamma = 3 \sin \alpha$.
14. If $b \cos \gamma = c \cos \beta$, then the triangle is isosceles.
15. If $b^2 + 2ac \cos \beta = a^2 + 2bc \cos \alpha$, then the triangle is isosceles.
16. If $a = 2b \cos \gamma$, then the triangle is isosceles.
17. If $a \cos \alpha = b \cos \beta$, then the triangle is isosceles or has a right angle.
18. If $\cos \beta + \cos \gamma = \frac{b+c}{a}$, then the triangle has a right angle.
19. In any triangle, $\frac{\cos \alpha}{a} + \frac{\cos \beta}{b} + \frac{\cos \gamma}{c} = \frac{a^2 + b^2 + c^2}{2abc}$.
20. In any triangle, $(a+b) \cos \gamma + (b+c) \cos \alpha + (c+a) \cos \beta = a + b + c$.

21. (Stewart's theorem) Let M be any point on side BC . Then $(\overline{AM})^2 = \frac{b^2 \cdot \overline{MB} + c^2 \cdot \overline{MC}}{a} - \overline{MB} \cdot \overline{MC}$.
22. In any triangle, $m_a^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4}$.
23. In any triangle, $i_a^2 = bc \left(1 - \left(\frac{a}{b+c} \right)^2 \right)$.
24. In any triangle, $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4} (a^2 + b^2 + c^2)$.
25. In any triangle, $m_a^4 + m_b^4 + m_c^4 = \frac{9}{16} (a^4 + b^4 + c^4)$.
26. In any triangle, if $b^2 + c^2 = 2a^2$, then $m_b^2 + m_c^2 = 2m_a^2$ and $cm_c = bm_b$.
27. In any triangle, if $\alpha \neq 90^\circ$, then $8A = (4m_a^2 - a^2) \tan \alpha$.
28. If $i_a = \frac{\sqrt{2} \cdot bc}{b+c}$, then the triangle has a right angle.
29. If $\frac{b+c}{a} = \frac{h_a}{i_a} \sqrt{2}$, then the triangle has a right angle.
30. If $\frac{b^3 + c^3 - a^3}{b+c-a} = a^2$, then $\alpha = 60^\circ$.
31. If $\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}$, then $\beta = 60^\circ$.
32. If $c(a+b-c) - a(b+c-a) + b(a+c-b) = bc$, then $\alpha = 60^\circ$.
33. Find the exact value of the area of the triangle in which $a = \sqrt{6}$, $\alpha = 60^\circ$, and $b+c = 3 + \sqrt{3}$.
34. If $(b+c)a = b^2 + c^2$ and $b+c = \frac{2\sqrt{3}}{3}a$, then $\alpha = 30^\circ$.
35. If $b^2 - c^2 = 2a^2$, then $\alpha \leq 30^\circ$.
36. If $b^2 + c^2 = 2a^2$, then $\alpha \leq 60^\circ$.
37. If $a < \frac{b+c}{2}$, then $\alpha < \frac{\beta+\gamma}{2}$.
38. If $b+c = 2a$ and $\beta+\gamma = 2\alpha$, then the triangle is equilateral.
39. In any triangle, $\frac{a}{b} = \frac{c + \cos \alpha (a \cos \gamma - c \cos \alpha)}{c \cos \gamma + a \cos \alpha}$.
40. In any triangle, $\sin^2 \alpha + 2 \sin \beta \sin \gamma \cos \alpha = \sin^2 \beta + \sin^2 \gamma$.
41. In any triangle, $4R^2 (\sin^2 \beta + \sin^2 \gamma - \sin^2 \alpha) = 2bc \cos \alpha$.
42. If $\sin \alpha = 2 \sin \beta \cos \gamma$, then the triangle is isosceles.
43. If $\sin \alpha = \frac{\sin \beta + \sin \gamma}{\cos \beta + \cos \gamma}$, then the triangle has a right angle.
44. In any triangle, $\frac{bc \cos \alpha + ac \cos \beta + ab \cos \gamma}{a \sin \alpha + b \sin \beta + c \sin \gamma} = R$.

45. In any triangle, $\frac{\sin \alpha + \sin \beta + \sin \gamma}{\cos \alpha + \cos \beta + \cos \gamma} = \frac{P}{2(R+r)}$.
46. If $E_1 = \frac{a^2 \sin \beta \cos \gamma}{\sin \alpha} + \frac{b^2 \cos \alpha \sin \gamma}{\sin \beta} + \frac{c^2 \sin \alpha \cos \beta}{\sin \gamma}$ and $E_2 = \frac{a^2 \cos \beta \sin \gamma}{\sin \alpha} + \frac{b^2 \sin \alpha \cos \gamma}{\sin \beta} + \frac{c^2 \cos \alpha \sin \beta}{\sin \gamma}$, then $E_1 = E_2$.
47. In any triangle, $\cot \alpha + \cot \beta + \cot \gamma = \frac{R(a^2 + b^2 + c^2)}{abc}$.
48. In any triangle, $a \cot \alpha + b \cot \beta + c \cot \gamma = 2(R+r)$.
49. In any triangle, $a \cos \alpha + b \cos \beta + c \cos \gamma = \frac{2A}{R}$.
50. In any triangle, $\cos \alpha + \cos \beta + \cos \gamma = 1 + \frac{r}{R}$.
51. In any triangle, $16A^2 = 2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$.
52. If $b^2 + c^2 = 2a^2$, then $8A = \sqrt{16b^2c^2 - (b^2 + c^2)^2}$.
53. If $\alpha \neq 90^\circ$, $\beta \neq 90^\circ$, and $a^2 \tan \beta = b^2 \tan \alpha$, then the triangle is isosceles or has a right angle.
54. If $\alpha \neq 90^\circ$, then $(b^2 + c^2 - a^2) \tan \alpha = 4A$.
55. If $\alpha \neq 90^\circ$, $\beta \neq 90^\circ$, then $(b^2 + c^2 - a^2) \tan \alpha = (a^2 + c^2 - b^2) \tan \beta$.
56. In any triangle, $a(b \cos \beta + c \cos \gamma) = \sin \alpha (b^2 \cot \beta + c^2 \cot \gamma)$.
57. In any triangle, $h_a = 2R \sin \beta \sin \gamma$.
58. In any triangle, $a^2 \cot \alpha + b^2 \cot \beta + c^2 \cot \gamma = 4A$.
59. If $2h_a = a \tan \beta$ ($\beta \neq 90^\circ$), then the triangle is isosceles.
60. If $\cot \alpha + \cot \beta = 2 \cot \gamma$, then $a^2 + b^2 = 2c^2$.
61. If $\cot \alpha = 2(c \cot \beta + c \cot \gamma)$, then $b^2 + c^2 = 3a^2$.
62. If $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, then the triangle has a right angle.
63. If $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$, then the triangle has a right angle.
64. If $a^2 + b^2 + c^2 = 8R^2$, then the triangle has a right angle.
65. If $b + c = 2a$, then $2 \cos \alpha + \cos \beta + \cos \gamma = 2$.
66. If $5a^2 = 5b^2 + c^2$ and $3b^2 = 3c^2 + a^2$, then $\tan \alpha = 3$, $\tan \beta = 2$, and $\tan \gamma = 1$.
67. If $\frac{\sin \gamma}{\sin \beta} + \frac{\sin \beta}{\sin \gamma} = 2 \sin \alpha$, then the triangle is isosceles with a right angle.
68. If $A = \frac{a^2 + b^2}{4}$, then the triangle is isosceles with a right angle.
69. What can we state about a triangle in which $\cos \alpha + \cos \beta + \cos \gamma = 1$?
70. If $P = b \cos \alpha + c \cos \beta + a \cos \gamma$, then the triangle is equilateral.