

Sample Problems

1. I am thinking of an angle α . If twice α is co-terminal to 100° , does that mean that α is co-terminal to 50° ?
2. Three times an angle β is co-terminal to 120° . Then β is co-terminal to what angle?
3. Consider the equation $\sin 2x = \frac{1}{2}$.
 - a) Solve the equation and present all solutions in degrees.
 - b) Find all solutions of the equation that fall between 0° and 360° .
 - c) Draw a picture of the solutions between 0° and 360° .
4. Consider the equation $\cos 3x = -\frac{\sqrt{3}}{2}$.
 - a) Find all solutions for the equation.
 - b) Find all solutions for the equation that fall between 0° and 360° . Present these angles in degrees.
 - c) Draw a picture of the solutions between 0° and 360° .
5. Consider the equation $\tan 5x = -1$.
 - a) Find all solutions for the equation. Present your answer in degrees.
 - b) Find all solutions for the equation. Present your answer in radians.
 - c) Find all solutions for the equation that fall between 0° and 360° .
 - d) Draw a picture of the solutions between 0° and 360° .
6.
 - a) Solve the equation $-\sin 5x = \cos 10x$
 - b) List all solutions (in degrees) that fall between 0° and 360° .

Practice Problems

1. Consider the equation $\sin 3x = -\frac{\sqrt{3}}{2}$.
 - a) Solve the equation. Present all solutions in degrees.
 - b) Solve the equation. Present all solutions in radians.
 - c) List all solutions between 0° and 360° .
 - d) Draw a picture of the solutions between 0° and 360° .
2. Find the exact value of all solutions for each of the following equations. Present your answer in radians.
 - a) $\tan 6x = -\frac{1}{\sqrt{3}}$
 - b) $\sin 3x = \cos 3x$
 - c) $\sin 4x = -1$
3. List all solutions of the equation $\sin 3x = -\frac{1}{2}$ that are between 0° and 360° .

Sample Problems - Answers

1. Not necessarily. α is co-terminal to either 50° or to 230° .

2. β is co-terminal to either 40° , or 160° , or 280°

3. a) $x = 15^\circ + k \cdot 180^\circ$
or $x = 75^\circ + k \cdot 180^\circ$
where $k \in \mathbb{Z}$

b) $15^\circ, 75^\circ, 195^\circ$, and 255°

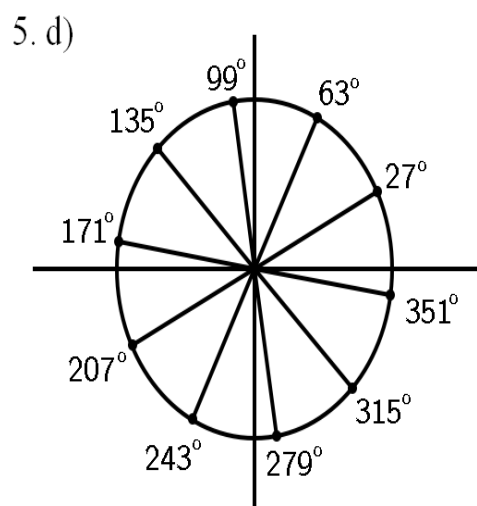
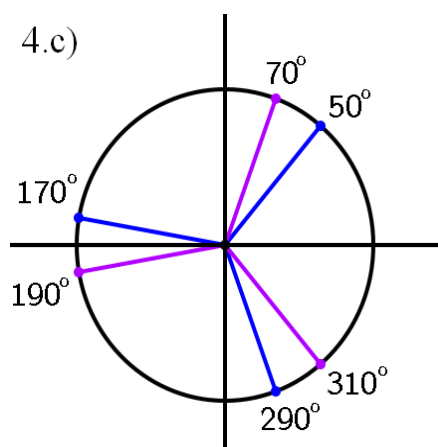
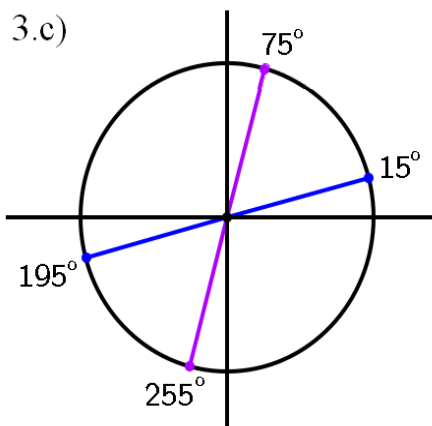
4. a) in degrees: $x = \pm 50^\circ + k \cdot 120^\circ$, where $k \in \mathbb{Z}$
in radians: $x = \pm \frac{5\pi}{18} + k \frac{2\pi}{3}$, $k \in \mathbb{Z}$

b) $50^\circ, 70^\circ, 170^\circ, 190^\circ, 290^\circ, 310^\circ$

5. a) $x = -9^\circ + k \cdot 36^\circ$ where $k \in \mathbb{Z}$

b) $x = -\frac{\pi}{20} + k \frac{\pi}{5}$ $k \in \mathbb{Z}$

c) $27^\circ, 63^\circ, 99^\circ, 171^\circ, 207^\circ,$
 $243^\circ, 279^\circ, 315^\circ$, and 351°



6. a) in degrees: $x = -6^\circ + k \cdot 72^\circ$ or $x = -25^\circ + k \cdot 72^\circ$ or $x = -18^\circ + k \cdot 72^\circ$ where $k \in \mathbb{Z}$

in radians: $x = -\frac{\pi}{30} + k \cdot \frac{2\pi}{5}$ or $x = -\frac{5\pi}{36} + k \cdot \frac{2\pi}{5}$ or $x = -\frac{\pi}{10} + k \cdot \frac{2\pi}{5}$ where $k \in \mathbb{Z}$

b) $47^\circ, 54^\circ, 66^\circ, 119^\circ, 126^\circ, 138^\circ, 191^\circ, 198^\circ, 210^\circ, 263^\circ, 270^\circ, 282^\circ, 335^\circ, 342^\circ, 354^\circ$

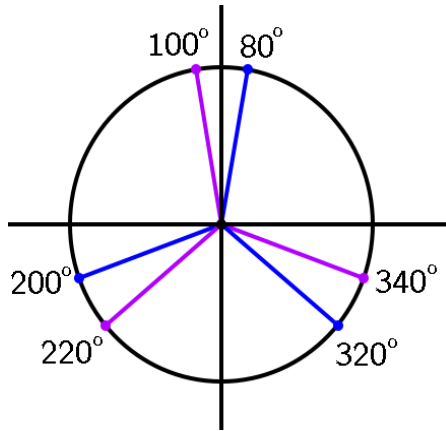
Practice Problems - Answers

1. a) $x = -20^\circ + k \cdot 120^\circ$ or $x = -40^\circ + k \cdot 120^\circ$ where $k \in \mathbb{Z}$

b) $x = -\frac{\pi}{9} + k \cdot \frac{2\pi}{3}$ or $x = -\frac{2\pi}{9} + k \cdot \frac{2\pi}{3}$ where $k \in \mathbb{Z}$

c) $80^\circ, 100^\circ, 200^\circ, 220^\circ, 320^\circ, 340^\circ$

d)



2. a) $-\frac{\pi}{36} + k\frac{\pi}{6}$ where $k \in \mathbb{Z}$ b) $\frac{\pi}{12} + k\frac{\pi}{3}$, $k \in \mathbb{Z}$ c) $-\frac{\pi}{8} + k\frac{\pi}{2}$, $k \in \mathbb{Z}$

3. $70^\circ, 110^\circ, 190^\circ, 230^\circ, 310^\circ, 350^\circ$

Sample Problems - Solutions

1. I am thinking of an angle α . If twice α is co-terminal to 100° , does that mean that α is co-terminal to 50° ?

Solution: Not necessarily. Let us express that twice α is co-terminal to 100° . Two angles are co-terminal when they differ by a multiple of 360° :

$$\begin{aligned} 2\alpha &= 100^\circ + k \cdot 360^\circ && \text{where } k \text{ is an integer} && \text{divide by 2} \\ \alpha &= 50^\circ + k \cdot 180^\circ \end{aligned}$$

Let us investigate what angles we obtained with the expression $50^\circ + k \cdot 180^\circ$.

If $k = 0$, then $\alpha = 50^\circ + 0 \cdot 180^\circ = 50^\circ$.

If $k = 1$, then $\alpha = 50^\circ + 1 \cdot 180^\circ = 230^\circ$.

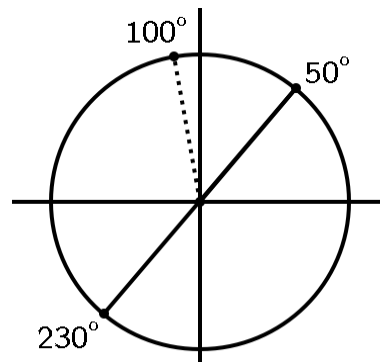
If $k = 2$, then $\alpha = 50^\circ + 2 \cdot 180^\circ = 410^\circ$ - co-terminal to 50° .

If $k = 3$, then $\alpha = 50^\circ + 3 \cdot 180^\circ = 590^\circ$ - co-terminal to 230° .

and so on, all such values are co-terminal to either 50° or to 230° .

Could α be 230° ? If α is 230° , then twice α is 460° which is indeed co-terminal to 100° since $460^\circ = 100^\circ + 360^\circ$.

So the answer is that α is either co-terminal to 50° or to 230° . These two angles are not co-terminal, they differ by 180° . But if we double them both, the difference between them becomes 360° - so their doubles are co-terminal.



2. Three times an angle β is co-terminal to 120° . Then β is co-terminal to what angle?

Solution: We state that three times β is co-terminal to 120° and divide both sides by 3.

$$\begin{aligned} 3\beta &= 120^\circ + k \cdot 360^\circ && \text{where } k \text{ is an integer} \\ \beta &= 40^\circ + k \cdot 120^\circ \end{aligned}$$

When $k = 0$, then $\beta = 40^\circ$.

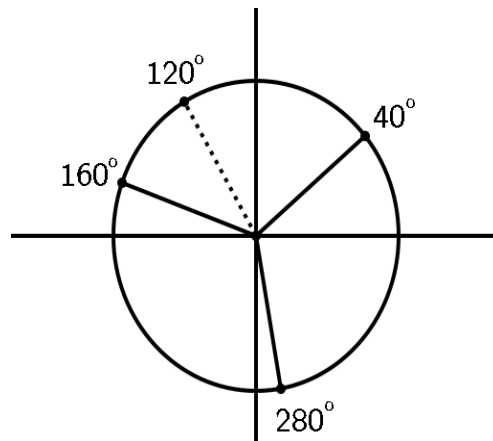
When $k = 1$, then $\beta = 160^\circ$.

When $k = 2$, then $\beta = 280^\circ$.

When $k = 3$, then $\beta = 400^\circ$ - co-terminal to 40° .

When $k = 4$, then $\beta = 520^\circ$ - co-terminal to 160° .

And so on, all other values will produce angles co-terminal with one of the angles above. So our answer is that β is co-terminal to either 40° , or 160° , or 280° .



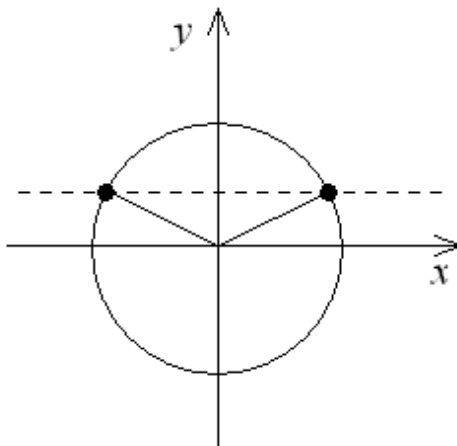
Does our answer make sense? We could argue that if two angles differ by 120° , then after multiplying both by 3, the difference becomes 360° - so they become co-terminal.

3. Consider the equation $\sin 2x = \frac{1}{2}$.

a) Solve the equation and present all solutions in degrees.

Solution: We will first solve for $2x$.

$$\sin 2x = \frac{1}{2}$$



$$2x = 30^\circ + k \cdot 360^\circ \quad \text{or} \quad 2x = 150^\circ + k \cdot 360^\circ \quad \text{where } k \in \mathbb{Z}$$

Next we solve for x in both equations by dividing both sides by 2.

$$x = 15^\circ + k \cdot 180^\circ \quad \text{or} \quad x = 75^\circ + k \cdot 180^\circ \quad \text{where } k \in \mathbb{Z}$$

b) Find all solutions of the equation that fall between 0° and 360° .

In this case, the picture above contains all these angles. To obtain these angles, we need to consider all suitable integer values of k in the expressions $x = 15^\circ + k \cdot 180^\circ$ and $x = 75^\circ + k \cdot 180^\circ$. First consider $x = 15^\circ + k \cdot 180^\circ$.

k	-1	0	1	2	3	4	5
$x = 15^\circ + k \cdot 180^\circ$	-165°	15°	195°	375°	555°	735°	915°

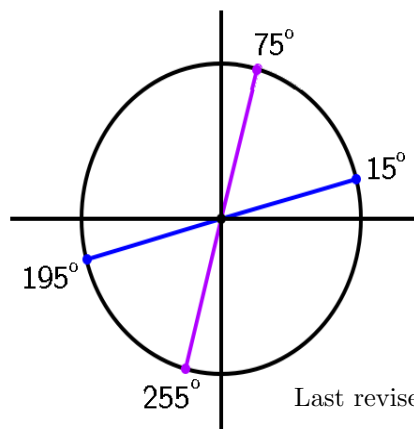
From all these values, 15° and 195° fall between 0° and 360° . Similarly, we consider $x = 75^\circ + k \cdot 180^\circ$

k	-1	0	1	2	3	4	5
$x = 75^\circ + k \cdot 180^\circ$	-105°	75°	255°	435°	615°	795°	975°

From all these values, 75° and 255° fall between 0° and 360° . So the final answer is 15° , 75° , 195° , and 255° .

c) Draw a picture of the solutions between 0° and 360° .

Solution: The first group, $x = 15^\circ + k \cdot 180^\circ$ (k integer) produces angles that are co-terminal with 15° or 195° . The second group, $x = 75^\circ + k \cdot 180^\circ$ (k integer) produces angles that are co-terminal with 75° or 255° .



4. Consider the equation $\cos 3x = -\frac{\sqrt{3}}{2}$.

a) Find all solutions for the equation.

Solution: We first solve for $3x$.

$$\begin{aligned}\cos 3x &= -\frac{\sqrt{3}}{2} \\ 3x &= \pm 150^\circ + k \cdot 360^\circ \text{ where } k \in \mathbb{Z} \quad \text{divide by 3}\end{aligned}$$

We now solve for x by dividing both sides by 3.

$$x = \pm 50^\circ + k \cdot 120^\circ \text{ where } k \text{ is an integer}$$

Finally, we convert the answer to radians

$$\begin{aligned}x &= \pm 50^\circ \left(\frac{\pi}{180^\circ} \right) + k \cdot 120^\circ \left(\frac{\pi}{180^\circ} \right) \text{ where } k \text{ is an integer} \\ x &= \pm \frac{5\pi}{18} + k \cdot \frac{2\pi}{3} \text{ where } k \text{ is an integer.}\end{aligned}$$

b) Find all solutions for the equation that fall between 0° and 360° . Present these angles in degrees.

Solution: Consider the general solution, $x = \pm 50^\circ + k \cdot 120^\circ$ where k is an integer. To obtain these angles, we need to consider all suitable integer values of k in the expressions $x = 50^\circ + k \cdot 120^\circ$ and $x = -50^\circ + k \cdot 120^\circ$. First consider $x = 50^\circ + k \cdot 120^\circ$.

k	-1	0	1	2	3	4	5
$x = 50^\circ + k \cdot 120^\circ$	-70°	50°	170°	290°	410°	530°	650°

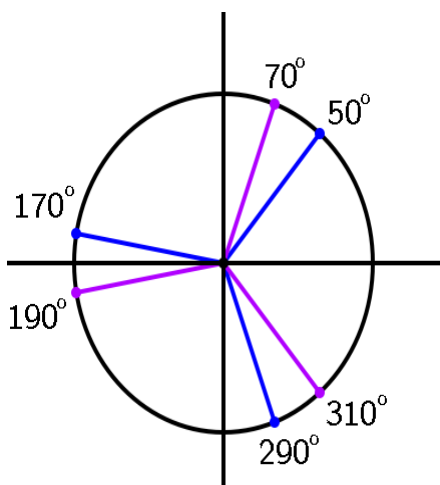
From all these values, 50° , 170° , and 290° fall between 0° and 360° . Similarly, we consider $x = -50^\circ + k \cdot 120^\circ$.

k	-1	0	1	2	3	4	5
$x = -50^\circ + k \cdot 120^\circ$	-170°	-50°	70°	190°	310°	430°	550°

From all these values, 70° , 190° and 310° fall between 0° and 360° . So the final answer is 50° , 70° , 170° , 190° , 290° and 310° .

c) Draw a picture of the solutions between 0° and 360° .

The group $x = 50^\circ + k \cdot 120^\circ$ (k integer) produces 50° , 170° , and 290° and the group $x = -50^\circ + k \cdot 120^\circ$ (k integer) produces 70° , 190° and 310° .



5. Consider the equation $\tan 5x = -1$.

a) Find all solutions for the equation. Present your answer in degrees.

Solution: We first solve for $5x$. Recall that the period of tangent is π and not 2π .

$$\begin{aligned}\tan 5x &= -1 \\ 5x &= -45^\circ + k \cdot 180^\circ \quad \text{where } k \in \mathbb{Z} && \text{divide by 5} \\ x &= -9^\circ + k \cdot 36^\circ \quad \text{where } k \in \mathbb{Z}\end{aligned}$$

b) Find all solutions for the equation. Present your answer in radians.

Solution: We could just convert the answer from part a). Or, we can solve the equation in radians.

$$\begin{aligned}\tan 5x &= -1 \\ 5x &= -\frac{\pi}{4} + k\pi \quad \text{where } k \in \mathbb{Z} && \text{divide by 5} \\ x &= -\frac{\pi}{20} + k\frac{\pi}{5} \quad k \in \mathbb{Z}\end{aligned}$$

c) Find all solutions for the equation that fall between 0° and 360° .

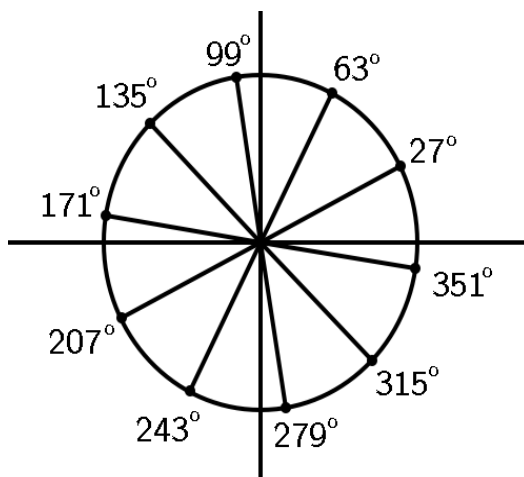
Solution: Substitute values into k starting with zero, and stopping once the solutions fall beyond 360° .

k	0	1	2	3	4	5	6	7	8	9	10	11
x (rad)	$-\frac{\pi}{20}$	$\frac{3\pi}{20}$	$\frac{7\pi}{20}$	$\frac{11\pi}{20}$	$\frac{15\pi}{20}$	$\frac{19\pi}{20}$	$\frac{23\pi}{20}$	$\frac{27\pi}{20}$	$\frac{31\pi}{20}$	$\frac{35\pi}{20}$	$\frac{39\pi}{20}$	$\frac{43\pi}{20}$
x (deg)	-9°	27°	63°	99°	135°	171°	207°	243°	279°	315°	351°	387°

So the solutions are: $27^\circ, 63^\circ, 99^\circ, 135^\circ, 171^\circ, 207^\circ, 243^\circ, 279^\circ, 315^\circ$, and 351° .

d) Draw a picture of the solutions between 0° and 360° .

Solution:



6. a) Solve the equation $-\sin 5x = \cos 10x$

b) List all solutions (in degrees) that fall between 0° and 360° .

Solution: Let us notice that 10 is twice 5 and so the double angle formula for cosine might be used. Let us denote $5x$ by B .

$$-\sin B = \cos 2B$$

We will use the double-angle formula for cosine. This formula has three forms, we will use the one that expresses things in terms of sine.

$$-\sin B = 1 - 2\sin^2 B$$

The equation is quadratic in $\sin B$. We solve for $\sin B$.

$$\begin{aligned} 2\sin^2 B - \sin B - 1 &= 0 \\ (2\sin B + 1)(\sin B - 1) &= 0 \\ \sin B = -\frac{1}{2} \quad \text{or} \quad \sin B &= 1 \end{aligned}$$

We now solve for B .

$$\sin B = -\frac{1}{2} \qquad \text{or} \qquad \sin B = 1$$

$$\begin{aligned} B &= -30^\circ + k \cdot 360^\circ & \text{or} & & B &= -90^\circ + k \cdot 360^\circ & \text{where } k \in \mathbb{Z} \\ B &= -150^\circ + k \cdot 360^\circ \end{aligned}$$

Recall that $B = 5x$.

$$\begin{aligned} 5x &= -30^\circ + k \cdot 360^\circ & \text{or} & & 5x &= -90^\circ + k \cdot 360^\circ & \text{where } k \in \mathbb{Z} \\ 5x &= -150^\circ + k \cdot 360^\circ \end{aligned}$$

We solve for x by dividing both sides by 5.

$$\begin{aligned} x &= -6^\circ + k \cdot 72^\circ & \text{or} & & x &= -18^\circ + k \cdot 72^\circ & \text{where } k \in \mathbb{Z} \\ x &= -25^\circ + k \cdot 72^\circ \end{aligned}$$

b) List all solutions (in degrees) that fall between 0° and 360° .

Solution: we start with the expression $-6^\circ + k \cdot 72^\circ$ (where $k \in \mathbb{Z}$) and substitute $k = 0, 1, 2, 3$, and 4. We obtain the angles

$$-6^\circ, 66^\circ, 138^\circ, 210^\circ, 282^\circ$$

Since -6° does not belong into the desired interval (between 0° and 360°), we need to replace that with a co-terminal angle that does. We can either add 360° to -6° or use $k = 5$ in the expression $-6^\circ + k \cdot 72^\circ$. Either way, we obtain 354° and so the list is

$$66^\circ, 138^\circ, 210^\circ, 282^\circ, 354^\circ$$

We apply the same method to the expressions $-25^\circ + k \cdot 72^\circ$ and obtain

$$-25^\circ, 47^\circ, 119^\circ, 191^\circ, 263^\circ \quad \text{and replace } -25^\circ \text{ with } 335^\circ$$

we apply the same method to the expression $-18^\circ + k \cdot 72^\circ$ and obtain

$$-18^\circ, 54^\circ, 126^\circ, 198^\circ, 270^\circ \quad \text{and replace } -18^\circ \text{ with } 342^\circ$$

So the complete list of all solutions between 0° and 360° is

$$47^\circ, 54^\circ, 66^\circ, 119^\circ, 126^\circ, 138^\circ, 191^\circ, 198^\circ, 210^\circ, 263^\circ, 270^\circ, 282^\circ, 335^\circ, 342^\circ, 354^\circ$$