

Sample Problems

Prove each of the following identities.

1. $\tan x \sin x + \cos x = \sec x$

2. $\frac{1}{\tan x} + \tan x = \frac{1}{\sin x \cos x}$

3. $\sin x - \sin x \cos^2 x = \sin^3 x$

4. $\frac{\cos \alpha}{1 + \sin \alpha} + \frac{1 + \sin \alpha}{\cos \alpha} = 2 \sec \alpha$

5. $\frac{\cos x}{1 - \sin x} - \frac{\cos x}{1 + \sin x} = 2 \tan x$

6. $\cos^2 x = \frac{\csc x \cos x}{\tan x + \cot x}$

7. $\frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1$

8. $\frac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$

9. $\frac{1 - \sin x}{\cos x} = \frac{\cos x}{1 + \sin x}$

10. $1 - 2 \cos^2 x = \frac{\tan^2 x - 1}{\tan^2 x + 1}$

11. $\tan^2 \theta = \csc^2 \theta \tan^2 \theta - 1$

12. $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$

13. $\frac{\csc \beta}{\sin \beta} - \frac{\cot \beta}{\tan \beta} = 1$

14. $\sin^4 x - \cos^4 x = 1 - 2 \cos^2 x$

15. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$

16. $\frac{\sin^2 x + 4 \sin x + 3}{\cos^2 x} = \frac{3 + \sin x}{1 - \sin x}$

17. $\frac{\cos x}{1 - \sin x} - \tan x = \sec x$

18. $\tan^2 x + 1 + \tan x \sec x = \frac{1 + \sin x}{\cos^2 x}$

Practice Problems

Prove each of the following identities.

$$1. \tan x + \frac{\cos x}{1 + \sin x} = \frac{1}{\cos x}$$

$$2. \tan^2 x + 1 = \sec^2 x$$

$$3. \frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} = 2 \tan x \sec x$$

$$4. \tan x + \cot x = \sec x \csc x$$

$$5. \frac{1 + \tan^2 x}{1 - \tan^2 x} = \frac{1}{\cos^2 x - \sin^2 x}$$

$$6. \tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$$

$$7. \frac{1 - \cos x}{\sin x} + \frac{\sin x}{1 - \cos x} = 2 \csc x$$

$$8. \frac{\sec x - 1}{\sec x + 1} = \frac{1 - \cos x}{1 + \cos x}$$

$$9. 1 + \cot^2 x = \csc^2 x$$

$$10. \frac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x$$

$$20. (\sin x - \tan x)(\cos x - \cot x) = (\sin x - 1)(\cos x - 1)$$

$$11. \frac{\cot x - 1}{\cot x + 1} = \frac{1 - \tan x}{1 + \tan x}$$

$$12. (\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x$$

$$13. \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$$

$$14. \frac{\cos x + 1}{\sin^3 x} = \frac{\csc x}{1 - \cos x}$$

$$15. \frac{1 + \sin x}{1 - \sin x} - \frac{1 - \sin x}{1 + \sin x} = 4 \tan x \sec x$$

$$16. \csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$$

$$17. \frac{\sin^2 x}{\cos^2 x + 3 \cos x + 2} = \frac{1 - \cos x}{2 + \cos x}$$

$$18. \frac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$$

$$19. \frac{1 + \tan x}{1 - \tan x} = \frac{\cos x + \sin x}{\cos x - \sin x}$$

Sample Problems - Solutions

1. $\tan x \sin x + \cos x = \sec x$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x .

$$\begin{aligned} \text{LHS} &= \tan x \sin x + \cos x = \frac{\sin x}{\cos x} \cdot \sin x + \cos x = \frac{\sin^2 x}{\cos x} + \cos x = \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} = \text{RHS} \end{aligned}$$

2. $\frac{1}{\tan x} + \tan x = \frac{1}{\sin x \cos x}$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x .

$$\text{LHS} = \frac{1}{\tan x} + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \text{RHS}$$

3. $\sin x - \sin x \cos^2 x = \sin^3 x$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x .

$$\text{LHS} = \sin x - \sin x \cos^2 x = \sin x (1 - \cos^2 x) = \sin x \cdot \sin^2 x = \text{RHS}$$

4. $\frac{\cos \alpha}{1 + \sin \alpha} + \frac{1 + \sin \alpha}{\cos \alpha} = 2 \sec \alpha$

Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x .

$$\begin{aligned} \text{LHS} &= \frac{\cos \alpha}{1 + \sin \alpha} + \frac{1 + \sin \alpha}{\cos \alpha} = \frac{\cos^2 \alpha}{(1 + \sin \alpha) \cos \alpha} + \frac{(1 + \sin \alpha)^2}{(1 + \sin \alpha) \cos \alpha} = \frac{\cos^2 \alpha + (1 + \sin \alpha)^2}{(1 + \sin \alpha) \cos \alpha} \\ &= \frac{\cos^2 \alpha + 1 + 2 \sin \alpha + \sin^2 \alpha}{(1 + \sin \alpha) \cos \alpha} = \frac{\cos^2 \alpha + \sin^2 \alpha + 1 + 2 \sin \alpha}{(1 + \sin \alpha) \cos \alpha} = \frac{2 + 2 \sin \alpha}{(1 + \sin \alpha) \cos \alpha} \\ &= \frac{2(1 + \sin \alpha)}{(1 + \sin \alpha) \cos \alpha} = \frac{2}{\cos \alpha} = 2 \cdot \frac{1}{\cos \alpha} = 2 \sec \alpha = \text{RHS} \end{aligned}$$

5. $\frac{\cos x}{1 - \sin x} - \frac{\cos x}{1 + \sin x} = 2 \tan x$

Solution: We will start with the left-hand side. First we bring the fractions to the common denominator. Recall that $\sin^2 x + \cos^2 x = 1$ for all values of x .

$$\begin{aligned} \text{LHS} &= \frac{\cos x}{1 - \sin x} - \frac{\cos x}{1 + \sin x} = \frac{\cos x (1 + \sin x)}{(1 - \sin x)(1 + \sin x)} - \frac{\cos x (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} \\ &= \frac{\cos x (1 + \sin x) - \cos x (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} = \frac{\cos x + \cos x \sin x - \cos x + \cos x \sin x}{1 - \sin^2 x} = \frac{2 \sin x \cos x}{\cos^2 x} \\ &= \frac{2 \sin x}{\cos x} = 2 \tan x = \text{RHS} \end{aligned}$$

$$6. \cos^2 x = \frac{\csc x \cos x}{\tan x + \cot x}$$

Solution: We will start with the right-hand side. We will re-write everything in terms of $\sin x$ and $\cos x$ and simplify. We will again run into the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned} \text{RHS} &= \frac{\csc x \cos x}{\tan x + \cot x} = \frac{\frac{1}{\sin x} \cdot \cos x}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{\frac{1}{\sin x} \cdot \frac{\cos x}{1}}{\frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x}} = \frac{\frac{\cos x}{\sin x}}{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x \cos x}} \\ &= \frac{\cos x}{\sin x} \cdot \frac{\cos x \sin x}{1} = \frac{\cos^2 x}{1} = \cos^2 x = \text{LHS} \end{aligned}$$

$$7. \frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1$$

Solution: We can factor the numerator via the difference of squares theorem.

$$\begin{aligned} \text{LHS} &= \frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = \frac{(\sin^2 x)^2 - (\cos^2 x)^2}{\sin^2 x - \cos^2 x} = \frac{(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{\sin^2 x - \cos^2 x} \\ &= \sin^2 x + \cos^2 x = 1 = \text{RHS} \end{aligned}$$

$$8. \frac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$$

Solution:

$$\begin{aligned} \text{LHS} &= \frac{\tan^2 x}{\tan^2 x + 1} = \frac{\left(\frac{\sin x}{\cos x}\right)^2}{\left(\frac{\sin x}{\cos x}\right)^2 + 1} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + 1} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} \\ &= \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \frac{\sin^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{1} = \sin^2 x = \text{RHS} \end{aligned}$$

$$9. \frac{1 - \sin x}{\cos x} = \frac{\cos x}{1 + \sin x}$$

Solution:

$$\begin{aligned} \text{LHS} &= \frac{1 - \sin x}{\cos x} = \frac{1 - \sin x}{\cos x} \cdot 1 = \frac{1 - \sin x}{\cos x} \cdot \frac{1 + \sin x}{1 + \sin x} = \frac{(1 - \sin x)(1 + \sin x)}{\cos x (1 + \sin x)} = \frac{1 - \sin^2 x}{\cos x (1 + \sin x)} \\ &= \frac{\cos^2 x}{\cos x (1 + \sin x)} = \frac{\cos x}{1 + \sin x} = \text{RHS} \end{aligned}$$

$$10. 1 - 2 \cos^2 x = \frac{\tan^2 x - 1}{\tan^2 x + 1}$$

Solution:

$$\begin{aligned} \text{RHS} &= \frac{\tan^2 x - 1}{\tan^2 x + 1} = \frac{\frac{\sin^2 x}{\cos^2 x} - 1}{\frac{\sin^2 x}{\cos^2 x} + 1} = \frac{\frac{\sin^2 x}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{\frac{\sin^2 x - \cos^2 x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} \\ &= \frac{\sin^2 x - \cos^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{1} = \sin^2 x - \cos^2 x \\ &= (1 - \cos^2 x) - \cos^2 x = 1 - 2 \cos^2 x = \text{LHS} \end{aligned}$$

$$11. \tan^2 \theta = \csc^2 \theta \tan^2 \theta - 1$$

$$\begin{aligned} \text{RHS} &= \csc^2 \theta \tan^2 \theta - 1 = \frac{1}{\sin^2 \theta} \cdot \left(\frac{\sin \theta}{\cos \theta} \right)^2 - 1 = \frac{1}{\sin^2 \theta} \cdot \frac{\sin^2 \theta}{\cos^2 \theta} - 1 = \frac{1}{\cos^2 \theta} - 1 \\ &= \frac{1}{\cos^2 \theta} - \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1 - \cos^2 \theta}{\cos^2 \theta} = \frac{\sin^2 \theta}{\cos^2 \theta} = \left(\frac{\sin \theta}{\cos \theta} \right)^2 = \tan^2 \theta = \text{LHS} \end{aligned}$$

$$12. \sec x + \tan x = \frac{\cos x}{1 - \sin x}$$

Solution:

$$\begin{aligned} \text{RHS} &= \frac{\cos x}{1 - \sin x} = \frac{\cos x}{1 - \sin x} \cdot 1 = \frac{\cos x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} = \frac{\cos x (1 + \sin x)}{(1 - \sin x)(1 + \sin x)} \\ &= \frac{\cos x (1 + \sin x)}{1 - \sin^2 x} = \frac{\cos x (1 + \sin x)}{\cos^2 x} = \frac{1 + \sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \text{LHS} \end{aligned}$$

$$13. \frac{\csc \beta}{\sin \beta} - \frac{\cot \beta}{\tan \beta} = 1$$

Solution: We will start with the left-hand side. We will re-write everything in terms of $\sin \beta$ and $\cos \beta$ and simplify. We will again run into the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$ for all angles x .

$$\begin{aligned} \text{LHS} &= \frac{\csc \beta}{\sin \beta} - \frac{\cot \beta}{\tan \beta} = \frac{\frac{1}{\sin \beta}}{\frac{\sin \beta}{1}} - \frac{\frac{\cos \beta}{\sin \beta}}{\frac{\sin \beta}{\cos \beta}} = \frac{1}{\sin \beta} \cdot \frac{1}{\sin \beta} - \frac{\cos \beta}{\sin \beta} \cdot \frac{\cos \beta}{\sin \beta} = \frac{1}{\sin^2 \beta} - \frac{\cos^2 \beta}{\sin^2 \beta} \\ &= \frac{1 - \cos^2 \beta}{\sin^2 \beta} = \frac{(\sin^2 \beta + \cos^2 \beta) - \cos^2 \beta}{\sin^2 \beta} = \frac{\sin^2 \beta}{\sin^2 \beta} = 1 = \text{RHS} \end{aligned}$$

$$14. \sin^4 x - \cos^4 x = 1 - 2 \cos^2 x$$

Solution:

$$\begin{aligned} \text{LHS} &= \sin^4 x - \cos^4 x = (\sin^2 x)^2 - (\cos^2 x)^2 = (\sin^2 x + \cos^2 x) (\sin^2 x - \cos^2 x) \\ &= 1 \cdot (\sin^2 x - \cos^2 x) = (1 - \cos^2 x) - \cos^2 x = 1 - 2 \cos^2 x = \text{RHS} \end{aligned}$$

$$15. (\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$$

Solution:

$$\begin{aligned} \text{LHS} &= (\sin x - \cos x)^2 + (\sin x + \cos x)^2 \\ &= (\sin^2 x + \cos^2 x - 2 \sin x \cos x) + (\sin^2 x + \cos^2 x + 2 \sin x \cos x) = 2 \sin^2 x + 2 \cos^2 x \\ &= 2 (\sin^2 x + \cos^2 x) = 2 \cdot 1 = 2 = \text{RHS} \end{aligned}$$

$$16. \frac{\sin^2 x + 4 \sin x + 3}{\cos^2 x} = \frac{3 + \sin x}{1 - \sin x}$$

Solution:

$$\text{LHS} = \frac{\sin^2 x + 4 \sin x + 3}{\cos^2 x} = \frac{(\sin x + 1)(\sin x + 3)}{1 - \sin^2 x} = \frac{(\sin x + 1)(\sin x + 3)}{(1 + \sin x)(1 - \sin x)} = \frac{\sin x + 3}{1 - \sin x} = \text{RHS}$$

$$17. \frac{\cos x}{1 - \sin x} - \tan x = \sec x$$

Solution:

$$\begin{aligned} \text{LHS} &= \frac{\cos x}{1 - \sin x} - \tan x = \frac{\cos x}{1 - \sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin x(1 - \sin x)}{\cos x(1 - \sin x)} = \frac{\cos^2 x - \sin x + \sin^2 x}{\cos x(1 - \sin x)} \\ &= \frac{(\cos^2 x + \sin^2 x) - \sin x}{\cos x(1 - \sin x)} = \frac{1 - \sin x}{\cos x(1 - \sin x)} = \frac{1}{\cos x} = \text{RHS} \end{aligned}$$

$$18. \tan^2 x + 1 + \tan x \sec x = \frac{1 + \sin x}{\cos^2 x}$$

Solution:

$$\begin{aligned} \text{LHS} &= \tan^2 x + 1 + \tan x \sec x = \frac{\sin^2 x}{\cos^2 x} + 1 + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \\ &= \frac{\sin^2 x + \cos^2 x + \sin x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x} = \text{RHS} \end{aligned}$$