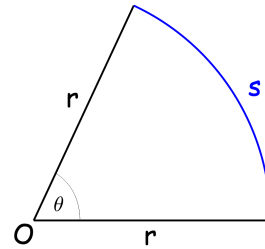


So far, angles have been measured in degrees, where one full revolution is divided into 360 equal parts. Degrees are a convenient, but ultimately arbitrary choice: there is nothing mathematically special about the number 360. In this section we introduce a different, more natural way to measure angles, called **radian measure**. Radian measure connects the size of an angle directly to the geometry of a circle, and it is the unit of choice throughout calculus and the sciences.

Definition: Let θ be any angle between 0° and 360° . Let us draw θ as a central angle in a circle of radius r , and let s denote the length of the arc subtended by θ . The measure of θ , in **radians**, is defined as the arc length divided by the radius:

$$\theta = \frac{s}{r}$$

The picture shown illustrates this definition. A circle of radius r is given, together with a central angle θ . The angle θ cuts off an arc of length s on the circle, and the radian measure of θ is the ratio $\frac{s}{r}$.



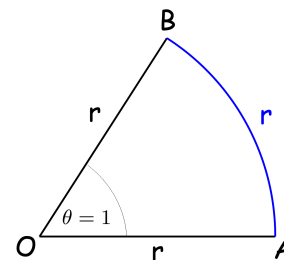
What kind of a unit is a radian? The definition is that it is a length divided by a length (meter over meter, or feet over feet), so the length units cancel out. Indeed, a radian is a unitless unit, so we will often see $\theta = 6$ instead of $\theta = 6 \text{ rad}$.

Consider now the full angle, one complete revolution around the circle. In this case, the arc subtended by θ is the entire circle, so its length is the circumference, $s = 2\pi r$. By the definition above, the radian measure of the full angle is

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$$

so one full revolution, 360° , measures 2π radians.

Next, consider a central angle θ for which the arc length s happens to equal the radius r itself. By the definition above, such an angle has radian measure $\theta = \frac{s}{r} = \frac{r}{r} = 1$: we call this a **radian**. The picture shown illustrates one radian. The two radii OA and OB have length r , and the arc from A to B also has length r . Because the arc length equals the radius, the central angle $\angle AOB$ measures exactly 1 radian, or approximately 57.3° .



Conversion between degrees and radians: One full revolution measures 360° and also 2π radians. Therefore

$$360^\circ = 2\pi \text{ radians} \quad \text{and} \quad 180^\circ = \pi \text{ radians}$$

To convert an angle from degrees to radians, multiply by $\frac{\pi}{180^\circ}$. To convert an angle from radians to degrees, multiply by $\frac{180^\circ}{\pi}$.

Example 1. Convert 135° to radians.

Solution: We multiply the angle by $\frac{\pi}{180^\circ}$.

$$135^\circ \cdot \frac{\pi}{180^\circ} = \frac{135\pi}{180} = \frac{3\pi}{4}$$

So 135° is the same angle as $\frac{3\pi}{4}$ radians.

Example 2. Convert $\frac{5\pi}{3}$ radians to degrees.

Solution: We multiply the angle by $\frac{180^\circ}{\pi}$.

$$\frac{5\pi}{3} \cdot \frac{180^\circ}{\pi} = \frac{5(180^\circ)}{3} = 5(60^\circ) = 300^\circ$$

So $\frac{5\pi}{3}$ radians is the same angle as 300° .

Now that we can convert angles back and forth between degrees and radians, we can actually deal with angles greater than 360° as well as negative angles, exactly as we do in degree measure. So the definition of radian measure can naturally be extended to all angles, not just those between 0° and 360° .

Theorem (Arc Length): Solving the definition $\theta = \frac{s}{r}$ for s , we obtain the arc length formula. Suppose a circle has radius r , and a central angle of measure θ , expressed in **radians**, is given. Then the length s of the arc subtended by that angle is

$$s = r\theta$$

Note that this formula only holds when θ is measured in radians; if the angle is given in degrees, it must first be converted.

Example 3. A circle has radius 10 cm and a central angle of 3 radians. Find the length of the arc subtended by this angle.

Solution: The angle is already given in radians, so we may apply the arc length formula $s = r\theta$ directly.

$$s = r\theta = (10 \text{ cm})(3) = 30 \text{ cm}$$



Sample Problems

1. Convert 150° to radians.
2. Convert $\frac{7\pi}{6}$ radians to degrees.
3. A circle has radius 12 cm. Find the length of the arc subtended by a central angle of 2 radians.
4. A wheel has radius 9 in. As the wheel turns through an angle of $\frac{3\pi}{4}$ radians, find the distance traveled by a point on the rim of the wheel.
5. An arc has length 15 m on a circle of radius 6 m. Find the central angle, in radians, subtended by this arc.
6. A satellite orbits the Earth at a distance of 7000 km from the center of the Earth. As the satellite moves through a central angle of 0.5 radians, how far does it travel?



Practice Problems

1. Convert 210° to radians.
2. Convert $\frac{5\pi}{4}$ radians to degrees.
3. A circle has radius 20 cm. Find the length of the arc subtended by a central angle of 1.5 radians.
4. A wheel has radius 10 in. As the wheel turns through an angle of $\frac{2\pi}{3}$ radians, find the distance traveled by a point on the rim of the wheel.
5. An arc has length 8 m on a circle of radius 4 m. Find the central angle, in radians, subtended by this arc.
6. The arm of a crane is 25 ft long. As the arm swings through an angle of 40° , how far does the tip of the arm travel?



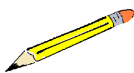
Answers

1. a) $\frac{5\pi}{6}$ b) 210° c) 24 cm d) $\frac{27\pi}{4}$ in ≈ 21.21 in e) 2.5 f) 3500 km

Practice Problems

1. a) $\frac{7\pi}{6}$ b) 225° c) 30 cm d) $\frac{20\pi}{3}$ in ≈ 20.94 in e) 2 f) $\frac{50\pi}{9}$ ft ≈ 17.45 ft

Sample Problems



Solutions

1. Convert 150° to radians.

Solution: We multiply the angle by $\frac{\pi}{180^\circ}$.

$$150^\circ \cdot \frac{\pi}{180^\circ} = \frac{150\pi}{180} = \frac{5\pi}{6}$$

So 150° is the same angle as $\frac{5\pi}{6}$ radians.

2. Convert $\frac{7\pi}{6}$ radians to degrees.

Solution: We multiply the angle by $\frac{180^\circ}{\pi}$.

$$\frac{7\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{7(180^\circ)}{6} = 7(30^\circ) = 210^\circ$$

So $\frac{7\pi}{6}$ radians is the same angle as 210° .

3. A circle has radius 12 cm. Find the length of the arc subtended by a central angle of 2 radians.

Solution: The angle is already given in radians, so we may apply the arc length formula $s = r\theta$ directly.

$$s = r\theta = (12 \text{ cm})(2) = 24 \text{ cm}$$

4. A wheel has radius 9 in. As the wheel turns through an angle of $\frac{3\pi}{4}$ radians, find the distance traveled by a point on the rim of the wheel.

Solution: The distance traveled by a point on the rim is exactly the arc length cut off by the given angle. Since the angle is already in radians, we apply $s = r\theta$.

$$s = r\theta = (9 \text{ in}) \left(\frac{3\pi}{4} \right) = \frac{27\pi}{4} \text{ in} \approx 21.21 \text{ in}$$

5. An arc has length 15 m on a circle of radius 6 m. Find the central angle, in radians, subtended by this arc.

Solution: We solve the arc length formula $s = r\theta$ for θ .

$$\begin{aligned}s &= r\theta \\ \theta &= \frac{s}{r} = \frac{15 \text{ m}}{6 \text{ m}} = 2.5\end{aligned}$$

The central angle measures 2.5 radians. Note that, since a radian is just a ratio of two lengths, it has no physical unit attached to it.

6. A satellite orbits the Earth at a distance of 7000 km from the center of the Earth. As the satellite moves through a central angle of 0.5 radians, how far does it travel?

Solution: Again, the angle is already given in radians, so the distance traveled is given directly by $s = r\theta$, where r is the distance from the center of the Earth to the satellite.

$$s = r\theta = (7000 \text{ km})(0.5) = 3500 \text{ km}$$