

Theorem: $\frac{d}{dx}(\ln x) = \frac{1}{x}$ for all $x > 0$

Proof: Let $x > 0$ be fixed.

$$\begin{aligned} \frac{d}{dx}(\ln x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \rightarrow 0} \frac{\ln \frac{x+h}{x}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \ln \frac{x+h}{x} \\ &= \lim_{h \rightarrow 0} \ln \left(\frac{x+h}{x} \right)^{1/h} = \lim_{h \rightarrow 0} \ln \left(1 + \frac{h}{x} \right)^{1/h} = \lim_{h \rightarrow 0} \ln \left(1 + \frac{1}{\frac{x}{h}} \right)^{\frac{1}{h}} = \lim_{h \rightarrow 0} \ln \left(1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h} \cdot \frac{1}{x}} \\ &= \lim_{h \rightarrow 0} \ln \left[\left(1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h}} \right]^{\frac{1}{x}} = \lim_{h \rightarrow 0} \left[\frac{1}{x} \cdot \ln \left(1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h}} \right] \end{aligned}$$

Since $\frac{1}{x}$ is a constant, we can write

$$\frac{d}{dx}(\ln x) = \lim_{h \rightarrow 0} \left(\frac{1}{x} \cdot \ln \left(1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h}} \right) = \frac{1}{x} \cdot \lim_{h \rightarrow 0} \ln \left(1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h}}$$

Define $m = \frac{x}{h}$. Since $x > 0$ is fixed, m approaches infinity as h approaches 0 from the right and negative infinity as h approaches 0 from the left. Then

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \lim_{h \rightarrow 0} \ln \left(1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h}} = \frac{1}{x} \lim_{m \rightarrow \infty} \ln \left(1 + \frac{1}{m} \right)^m$$

Since $f(x) = \ln x$ is continuous at e , we also have that

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \lim_{m \rightarrow \infty} \ln \left(1 + \frac{1}{m} \right)^m = \frac{1}{x} \ln \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m = \frac{1}{x} \ln e = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

Theorem: $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$ for all $a > 0$, $a \neq 1$

Proof: We will use the change-base theorem.

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx} \left(\frac{\ln x}{\ln a} \right) = \frac{d}{dx} \left(\frac{1}{\ln a} \cdot \ln x \right) = \frac{1}{\ln a} \frac{d}{dx}(\ln x) = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$$



Sample Problems

Differentiate each of the following.

1. $f(x) = \ln(x^5)$

2. $f(x) = \log_3 x$

3. $f(x) = x \ln x - x$

4. $f(x) = \frac{1}{2x^2} \left(\ln\left(\frac{1}{x}\right) - \frac{1}{2} \right)$

5. $f(x) = \ln(\sqrt[3]{2x^4})$



Answers

1. $f(x) = \ln(x^5)$

Solution: Recall the identity $\log_a(b^c) = c \log_a b$. Using that, $\ln(x^5) = 5 \ln x$.

$$f'(x) = \frac{d}{dx}(\ln(x^5)) = \frac{d}{dx}(5 \ln x) = 5 \frac{d}{dx}(\ln x) = 5 \cdot \frac{1}{x} = \boxed{\frac{5}{x}}$$

2. $f(x) = \log_3 x$

Solution: Recall the change base theorem: $\log_a b = \frac{\log_c b}{\log_c a}$. We use this to switch to natural logarithm. Using this identity, $\log_3 x = \frac{\ln x}{\ln 3}$ and since $\ln 3$ is just a constant, so is $\frac{1}{\ln 3}$ and so we reduced the situation to a constant multiplier.

$$f'(x) = \frac{d}{dx}(\log_3 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 3}\right) = \frac{1}{\ln 3} \cdot \frac{d}{dx}(\ln x) = \frac{1}{\ln 3} \cdot \frac{1}{x} = \boxed{\frac{1}{x \ln 3}}$$

3. $f(x) = x \ln x - x$

Solution: We use the product rule in differentiating $x \ln x$.

$$f'(x) = (x)' \ln x + x (\ln x)' - (x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 = \ln x + 1 - 1 = \boxed{\ln x}$$

This is an interesting result. We have stumbled into a function whose derivative is $\ln x$. Is there a more straightforward way to find this?

4. $f(x) = \frac{1}{2x^2} \left(\ln\left(\frac{1}{x}\right) - \frac{1}{2} \right)$

Solution: We re-write $\ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x$ via the identity $\log_a(b^c) = c \log_a b$. Then we distribute $\frac{1}{2x^2}$ and use the product rule.

$$\begin{aligned} f(x) &= \frac{1}{2x^2} \left(\ln\left(\frac{1}{x}\right) - \frac{1}{2} \right) = \frac{1}{2x^2} \left(-\ln x - \frac{1}{2} \right) = \frac{-\ln x}{2x^2} - \frac{1}{4x^2} = -\frac{1}{2} (\ln x) (x^{-2}) - \frac{1}{4} x^{-2} \\ f'(x) &= \left(-\frac{1}{2} (\ln x) (x^{-2}) - \frac{1}{4} x^{-2} \right)' = -\frac{1}{2} \left((\ln x)' x^{-2} + \ln x (x^{-2})' \right) - \frac{1}{4} (x^{-2})' \\ &= -\frac{1}{2} \left(\frac{1}{x} \cdot \frac{1}{x^2} - 2 \ln x (x^{-3}) \right) - \frac{1}{4} (-2) x^{-3} = -\frac{1}{2x^3} + \frac{\ln x}{x^3} + \frac{1}{2x^3} = \boxed{\frac{\ln x}{x^3}} \end{aligned}$$

$$5. f(x) = \ln \left(\sqrt[3]{2x^4} \right)$$

Solution: Recall that $\sqrt[3]{A} = A^{1/3}$ and so the radical can be made 'go away' just like the exponent. All complications can be resolved before the calculus just via the rules of logarithms. Recall the logarithmic rule

$$\log_a b + \log_a c = \log_a bc.$$

$$f(x) = \ln \left(\sqrt[3]{2x^4} \right) = \ln \left((2x^4)^{1/3} \right) = \frac{1}{3} \ln (2x^4) = \frac{1}{3} (\ln 2 + \ln (x^4)) = \frac{1}{3} \ln 2 + \frac{1}{3} \cdot 4 \ln x = \frac{\ln 2}{3} + \frac{4}{3} \ln x$$

$$f'(x) = \left(\frac{\ln 2}{3} + \frac{4}{3} \ln x \right)' = \left(\frac{\ln 2}{3} \right)' + \left(\frac{4}{3} \ln x \right)' = 0 + \frac{4}{3} \cdot \frac{1}{x} = \boxed{\frac{4}{3x}}$$

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