

Logic studies how the truth of some sentences depends on the truth of others. The sentences it takes as its raw material are *statements*: declarative sentences that are either true or false, but not both.

Definition. A **statement** (or **proposition**) is a declarative sentence that has exactly one **truth value**, either **true** (T) or **false** (F).

“The number 7 is prime” is a statement (a true one). “The number 8 is prime” is also a statement (a false one). “Close the door” and “What time is it?” are not statements, because they are not declarative.

A statement that contains no other statement as a part is called **simple** (or **atomic**). We abbreviate simple statements by capital letters $P, Q, R, \dots$, called **statement variables**. A statement built from simpler statements by means of words such as *and*, *or*, and *if–then* is called **compound**.

Definition. A **logical connective** is a word or phrase that combines one or more statements into a new statement whose truth value is completely determined by the truth values of the parts.

This section treats the three connectives that join two statements:

Name	English	Symbol	Read as
Conjunction	<i>and</i>	$P \wedge Q$	“ P and Q ”
Disjunction	<i>or</i>	$P \vee Q$	“ P or Q ”
Conditional	<i>if–then</i>	$P \rightarrow Q$	“if P , then Q ”

The tool we use to record the meaning of a connective is the **truth table**. It lists every possible combination of truth values for the simple parts and, in the final column, the resulting truth value of the compound statement. Two simple statements give $2 \times 2 = 4$ rows.

Conjunction: *and*

Definition. The **conjunction** of statements P and Q , written $P \wedge Q$, is the statement “ P and Q .” It is true exactly when *both* P and Q are true, and false otherwise.

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

A single false conjunct is enough to make the whole conjunction false.

Example 1 Let P be “12 is even” (true) and Q be “12 is a multiple of 3” (true). Then $P \wedge Q$, “12 is even and 12 is a multiple of 3,” is true. If instead Q were “12 is a multiple of 5” (false), the conjunction would be false, even though P is still true.

Disjunction: *or*

Definition: The **disjunction** of statements P and Q , written $P \vee Q$, is the statement “ P or Q .” It is true when *at least one* of P , Q is true, and false only when both are false.

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

Inclusive versus exclusive *or*. The connective $\vee$ is the **inclusive** “or”: it allows the case in which both parts are true (row 1 above). English *or* is usually inclusive — “The applicant must know French or German” certainly does not disqualify someone who knows both. Occasionally English uses the **exclusive** “or,” meaning “one or the other but not both,” as in “The set menu comes with soup or salad.” Unless a sentence clearly rules out the “both” case, translate *or* as the inclusive $\vee$.

Example 2 Let A be “ $x > 0$ ” and B be “ $x < 10$.” For $x = 5$ both are true, so A or B is true. For $x = 20$, A is true and B is false, so A or B is still true. In fact A or B is false for no real number x at all, since every x satisfies at least one of the two inequalities.

The Conditional: *if–then*

Definition: The **conditional** with **antecedent** P and **consequent** Q , written $P \rightarrow Q$, is the statement “if P , then Q .” It is false in the single case where P is true and Q is false, and true in every other case.

P	Q	if P , then Q
T	T	T
T	F	F
F	T	T
F	F	T

Why the last two rows are true. A conditional is a *promise* about what holds when the antecedent is met. Consider the pledge

“If you finish the assignment, then you may go out.”

The promise is broken only if you finish the assignment and are *still* kept in — that is, antecedent true, consequent false (row 2). If you do not finish the assignment (rows 3 and 4), the promise has not been broken, whatever happens next, so we count it as true. A conditional whose antecedent is false is said to be **vacuously true**. Thus “If $2 + 2 = 5$, then the moon is made of cheese” is a true statement.

Ways of saying *if–then*. The following English phrases all translate to $P \rightarrow Q$:

English	Symbolization
If P , then Q .	$P \rightarrow Q$
Q if P .	$P \rightarrow Q$
P implies Q .	$P \rightarrow Q$
P is sufficient for Q .	$P \rightarrow Q$
Q is necessary for P .	$P \rightarrow Q$
Q whenever P .	$P \rightarrow Q$

The biconditional. When both $P \rightarrow Q$ and its converse $Q \rightarrow P$ hold, we write $P \leftrightarrow Q$, read “ P if and only if Q ” (often abbreviated “ P iff Q ”). It is true exactly when P and Q have the same truth value.

Summary

P	Q	P and Q	P or Q	If P , then Q
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

- **Conjunction** P and Q : true only when both parts are true.
- **Disjunction** P or Q : false only when both parts are false (inclusive “or”).
- **Conditional** If P , then Q : false only when the antecedent is true and the consequent is false; vacuously true whenever the antecedent is false.



Practice Problems

- Classify each sentence as a statement or not. If it is a statement, give its truth value.
 - 5 is an odd number.
 - Is 5 an odd number?
 - $2 + 1 = 4$.
 - Every even number greater than 2 is a sum of two primes.
- Determine about each statement whether they are true or false.
 - 2 is a prime number and 5 is even.
 - 2 is a prime number or 5 is even.
 - If 2 is a prime number, then 5 is even.
 - If 5 is even, then 2 is a prime number.
 - $-3 \in \mathbb{Z}$ and the angles of a regular triangle are all 60° .
 - $-3 \in \mathbb{Z}$ or the angles of a regular triangle are all 60° .
 - If $-3 \in \mathbb{Z}$, then the angles of a regular triangle are all 60° .

- (h) If the angles of a regular triangle are all 60° , then $-3 \in \mathbb{Z}$.
 - (i) 7 is a triangular number and 8 is a prime.
 - (j) 7 is a triangular number or 8 is a prime.
 - (k) If 7 is a triangular number, then 8 is a prime.
 - (l) If 8 is a prime, then 7 is a triangular number.
3. Translate back into idiomatic English if P means "Sue is in a good mood", Q means "today is Friday", and R means "we are going out tonight."
- (a) $(P \wedge Q) \rightarrow R$
 - (b) $P \rightarrow (Q \vee R)$
4. Determine whether the given statements are true or false.
- (a) If a number is divisible by 2, then it is also divisible by 4.
 - (b) If a number is divisible by 6, then it is also divisible by 2.
 - (c) If a and b are integers and ab is divisible by 10, then a or b is divisible by 10.
 - (d) If a and b are integers and ab is divisible by 7, then a or b is divisible by 7.
5. Construct a full truth table for each formula.
- (a) $P \rightarrow (Q \text{ or } R)$
 - (b) $(P \text{ and } Q) \rightarrow P$
 - (c) $(P \rightarrow Q) \text{ and } (Q \rightarrow P)$



Answers

1. a) true statement b) not a statement c) false statement
d) statement, not yet proven or dis-proven (called the Goldbach conjecture)
2. a) false b) true c) false d) true e) true f) true g) true
h) true i) false j) false k) true l) true
3. a) If Sue is in a good mood, and today is Friday, then we are going out tonight.
b) If Sue is in a good mood, then it is Friday or we are going out tonight.
4. a) false b) true c) false d) true

5. a)

P	Q	R	If P , then Q or R
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

b)

P	Q	R	If P and Q , then R
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

c)

P	Q	If P , then Q , and if Q , then P
T	T	T
T	F	F
F	T	F
F	F	T

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