

The basic units of propositional logic are *sentences* (also called *statements* or *propositions*): declarative expressions that are either true or false, but not both. We assign each sentence exactly one of the two *truth values*, denoted T (true) and F (false).

**Definition:** An **atomic sentence** is a sentence that contains no other sentence as a proper part. Atomic sentences are represented by **sentence letters**  $p, q, r, \dots$  (possibly with subscripts).

**Definition:** A **compound sentence** is a sentence built from one or more shorter sentences by means of one or more **logical connectives**. The shorter sentences from which it is built are its **components**.

**Example 1** Let  $p$  abbreviate “It is raining” and  $q$  abbreviate “The match is canceled.” Then:

“It is raining” is atomic ( $p$ ).

“It is not raining” is compound: the connective *it is not the case that* is applied to  $p$ .

“It is raining and the match is cancelled” is compound, with components  $p$  and  $q$ .

A connective is **truth-functional** when the truth value of any compound sentence formed with it is completely determined by the truth values of its components. Propositional logic studies exactly the five truth-functional connectives below.

| Name          | Symbol            | English reading                  |
|---------------|-------------------|----------------------------------|
| Negation      | $\neg$            | “it is not the case that”, “not” |
| Conjunction   | $\wedge$          | “and”, “but”, “however”          |
| Disjunction   | $\vee$            | “or” (inclusive),                |
| Conditional   | $\rightarrow$     | “if ... then”, “only if”         |
| Biconditional | $\leftrightarrow$ | “if and only if”, ‘              |

Identifying the main connective is the single most important parsing skill: it tells us what *kind* of sentence we have. The sentence  $\neg(p \wedge q)$  is a *negation* (main connective  $\neg$ ), whereas  $\neg p \wedge q$  is a *conjunction* (main connective  $\wedge$ ), and the two are not equivalent.

## Truth Tables

We now give the meaning of each connective by stating, for every combination of truth values of the components, the truth value of the compound. These are the **characteristic truth tables**.

| $\phi$ | $\psi$ | $\neg\phi$ | $\phi \wedge \psi$ | $\phi \vee \psi$ | $\phi \rightarrow \psi$ | $\phi \leftrightarrow \psi$ |
|--------|--------|------------|--------------------|------------------|-------------------------|-----------------------------|
| T      | T      | F          | T                  | T                | T                       | T                           |
| T      | F      | F          | F                  | T                | F                       | F                           |
| F      | T      | T          | F                  | T                | T                       | F                           |
| F      | F      | T          | F                  | F                | T                       | T                           |

In words:

- $\neg\phi$  is true exactly when  $\phi$  is false.
- $\phi \wedge \psi$  is true exactly when *both* conjuncts are true.
- $\phi \vee \psi$  is true exactly when *at least one* disjunct is true. This is **inclusive** disjunction; it is also true when both disjuncts are true.
- $\phi \rightarrow \psi$  is false in the single case where the antecedent is true and the consequent is false; otherwise it is true. In particular it is **vacuously true** whenever the antecedent is false.
- $\phi \leftrightarrow \psi$  is true exactly when  $\phi$  and  $\psi$  have the *same* truth value.

**Remark:** The  $\rightarrow$  is called the **conditional**. It captures only the pattern “not: antecedent true and consequent false,” and deliberately ignores any connection of meaning, causation, or relevance between antecedent and consequent. Consequently “If  $2 + 2 = 5$ , then the Moon is made of cheese” counts as true. This is a known point of divergence from some uses of “if ... then” in ordinary language.

**Example 2.** A step-by-step table Consider  $(p \rightarrow q) \wedge \neg q$ . Its main connective is  $\wedge$ ; its immediate sub-formulas are  $p \rightarrow q$  and  $\neg q$ .

| $p$ | $q$ | $p \rightarrow q$ | $\neg q$ | $(p \rightarrow q) \wedge \neg q$ |
|-----|-----|-------------------|----------|-----------------------------------|
| T   | T   | T                 | F        | F                                 |
| T   | F   | F                 | T        | F                                 |
| F   | T   | T                 | F        | F                                 |
| F   | F   | T                 | T        | T                                 |

The compound is true only when  $p = F, q = F$ .

## Classifying Compound Sentences

**Definition:** Let  $\phi$  be a statement.

- $\phi$  is a **tautology** (*logically true*) if it is true on every assignment. For example,  $q \vee (\neg q)$  is always true.
- $\phi$  is a **contradiction** (*logically false*) if it is false on every assignment. For example,  $q \wedge (\neg q)$  is always false.
- $\phi$  is **contingent** if it is true on at least one assignment and false on at least one assignment.
- $\phi$  is **satisfiable** if it is true on at least one assignment.

**Example 3.**  $p \vee \neg p$  is a tautology;  $p \wedge \neg p$  is a contradiction;  $p \rightarrow q$  is contingent. The sentence  $(p \rightarrow q) \wedge \neg q$  from the previous example is contingent and satisfiable.

## Logical Equivalence

**Definition:** Statements  $\phi$  and  $\psi$  are **logically equivalent**, written  $\phi \equiv \psi$ , if they have the same truth value on every assignment; equivalently, if  $\phi \leftrightarrow \psi$  is a tautology.

Equivalences let us rewrite compound sentences into simpler or more convenient forms. The following hold for all wffs  $\phi, \psi, \chi$ .

|                 |                                                                                                                  |
|-----------------|------------------------------------------------------------------------------------------------------------------|
| Double negation | $\neg\neg\phi \equiv \phi$                                                                                       |
| Commutativity   | $\phi \wedge \psi \equiv \psi \wedge \phi; \quad \phi \vee \psi \equiv \psi \vee \phi$                           |
| Associativity   | $(\phi \wedge \psi) \wedge \chi \equiv \phi \wedge (\psi \wedge \chi)$                                           |
| Distributivity  | $\phi \wedge (\psi \vee \chi) \equiv (\phi \wedge \psi) \vee (\phi \wedge \chi)$                                 |
| De Morgan       | $\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$<br>$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$ |
| Conditional     | $\phi \rightarrow \psi \equiv \neg\phi \vee \psi$                                                                |
| Contrapositive  | $\phi \rightarrow \psi \equiv \neg\psi \rightarrow \neg\phi$                                                     |
| Biconditional   | $\phi \leftrightarrow \psi \equiv (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$                        |

## Translating Between English and Symbols

To symbolise an English sentence:

1. Identify the atomic sentences and assign each a sentence letter. Use the same letter for repeated occurrences of the same atomic sentence.
2. Identify the connective words and phrases, and determine the main connective from the sentence's structure and punctuation.
3. Build the wff outward from the atoms, adding parentheses to fix scope.

| English pattern                                                  | Symbolisation                                  |
|------------------------------------------------------------------|------------------------------------------------|
| not $A$ ; it is false that $A$                                   | $\neg A$                                       |
| $A$ but $B$ ; $A$ although $B$                                   | $A \wedge B$                                   |
| neither $A$ nor $B$                                              | $\neg A \wedge \neg B$                         |
| $A$ unless $B$                                                   | $\neg B \rightarrow A \quad (\equiv A \vee B)$ |
| if $A$ then $B$ ; $B$ if $A$ ; $A$ only if $B$                   | $A \rightarrow B$                              |
| $A$ provided that $B$                                            | $B \rightarrow A$                              |
| $A$ if and only if $B$ ; $A$ is necessary and sufficient for $B$ | $A \leftrightarrow B$                          |

**Example 4.** “If the alarm sounds and the door is unlocked, then either security responds or the manager is notified.”

Let  $a$ : the alarm sounds;  $d$ : the door is unlocked;  $s$ : security responds;  $m$ : the manager is notified.

Symbolisation:  $(a \wedge d) \rightarrow (s \vee m)$ .



## Practice Problems

1. For each formula, state its main connective and say whether it is a negation, conjunction, disjunction, conditional, or biconditional. Construct a truth table for each.
  - (a)  $\neg p \rightarrow (q \wedge r)$
  - (b)  $\neg(p \rightarrow (q \wedge r))$
  - (c)  $(p \leftrightarrow \neg q) \vee r$
2. Construct full truth tables for  $p \rightarrow (q \rightarrow p)$  and  $(p \rightarrow q) \rightarrow p$ . Classify each as tautology, contradiction, or contingent.
3. Show that  $p \rightarrow q$  and  $\neg q \rightarrow \neg p$  are logically equivalent, and that neither is equivalent to  $q \rightarrow p$ .
4. Symbolize, stating your dictionary:
  - (a) "The bridge is safe only if it has been inspected and certified."
  - (b) "Neither the report nor the summary was submitted, but the deadline was extended."
  - (c) "You may enter unless your name is not on the list."



## Answers

1. a) Conditional

| $p$ | $q$ | $r$ | $\neg p \rightarrow (q \wedge r)$ |
|-----|-----|-----|-----------------------------------|
| T   | T   | T   | T                                 |
| T   | T   | F   | T                                 |
| T   | F   | T   | T                                 |
| T   | F   | F   | T                                 |
| F   | T   | T   | T                                 |
| F   | T   | F   | F                                 |
| F   | F   | T   | F                                 |
| F   | F   | F   | F                                 |

b) Negation

| $p$ | $q$ | $r$ | $\neg(p \rightarrow (q \wedge r))$ |
|-----|-----|-----|------------------------------------|
| T   | T   | T   | F                                  |
| T   | T   | F   | T                                  |
| T   | F   | T   | T                                  |
| T   | F   | F   | T                                  |
| F   | T   | T   | F                                  |
| F   | T   | F   | F                                  |
| F   | F   | T   | F                                  |
| F   | F   | F   | F                                  |

c) Disjunctive

| $p$ | $q$ | $r$ | $(p \leftrightarrow \neg q) \vee r$ |
|-----|-----|-----|-------------------------------------|
| T   | T   | T   | T                                   |
| T   | T   | F   | F                                   |
| T   | F   | T   | T                                   |
| T   | F   | F   | T                                   |
| F   | T   | T   | T                                   |
| F   | T   | F   | T                                   |
| F   | F   | T   | T                                   |
| F   | F   | F   | F                                   |

2. a) tautology

| $p$ | $q$ | $p \rightarrow (q \rightarrow p)$ |
|-----|-----|-----------------------------------|
| T   | T   | T                                 |
| T   | F   | T                                 |
| F   | T   | T                                 |
| F   | F   | T                                 |

b) contingent

| $p$ | $q$ | $(p \rightarrow q) \rightarrow p$ |
|-----|-----|-----------------------------------|
| T   | T   | T                                 |
| T   | F   | T                                 |
| F   | T   | F                                 |
| F   | F   | F                                 |

3.  $p \rightarrow q$  and  $\neg q \rightarrow \neg p$  have the same truth tables,  $q \rightarrow p$  doesn't.

| $p$ | $q$ | $p \rightarrow q$ |
|-----|-----|-------------------|
| T   | T   | T                 |
| T   | F   | F                 |
| F   | T   | T                 |
| F   | F   | T                 |

| $p$ | $q$ | $\neg q \rightarrow \neg p$ |
|-----|-----|-----------------------------|
| T   | T   | T                           |
| T   | F   | F                           |
| F   | T   | T                           |
| F   | F   | T                           |

| $p$ | $q$ | $q \rightarrow p$ |
|-----|-----|-------------------|
| T   | T   | T                 |
| T   | F   | T                 |
| F   | T   | F                 |
| F   | F   | T                 |

4. a)  $p$  = The bridge is safe.  $q$  = The bridge has been inspected.  $r$  = The bridge has been certified.

Then the statement is  $(q \wedge r) \rightarrow p$

b)  $p$  = The report was submitted.  $q$  = The summary was submitted.  $r$  = The deadline was extended.

Then the statement is  $(\neg p \wedge \neg q) \wedge r$

c)  $p$  = You may enter.  $q$  = Your name is on the list.  $r$  = The deadline was extended.

Then the statement is  $q \rightarrow p$

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